TEX macros for proof boxes

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1 Introduction

The proof

1	$\alpha \leftrightarrow \psi(x,\top)$		
2	$\exists\beta.\,\psi(x,\beta)$	total	
$\exists\beta$ 3	$\psi(x,\beta)$		
4	β		α
5	$\beta = \top$ $\quad(*)$		$\psi(x,\top)$ $\quad\leftrightarrow\mathcal{E}(1)$
6	$\psi(x,\top)$ $\quad$ subs		$\beta = \top$ $\quad$ func
7	α $\quad\leftrightarrow\mathcal{E}(1)$		β $\quad(*)$
8	$\alpha = \beta$	$\leftrightarrow\mathcal{I}$	
9	$\psi(x,\alpha)$	subs	
10	$\psi(x,\alpha)$	$\exists\mathcal{E}$	

$\psi(x,\alpha)$		
α		$\psi(x,\top)$
$\alpha = \top$ $\quad(*)$		$\alpha = \top$ $\quad$ func
$\phi(x,\top)$ $\quad$ subs		α $\quad(*)$
$\alpha \leftrightarrow \psi(x,\top)$	$\leftrightarrow\mathcal{I}$	

11 $\quad \psi(x,\alpha) \leftrightarrow (\alpha \leftrightarrow \psi(x,\top))$ $\qquad \leftrightarrow\mathcal{I}$

is produced by

```
\begin{proofbox}
  \(\"1"\:\alpha\leftrightarrow\psi(x,\top)\\
        \:\Some\beta.\psi(x,\beta)                      \=\mathsf{total}\\
    \[\exists\beta\kern-1em\:\psi(x,\beta)\\
      \(\:\beta\\
        \:\beta=\top                                    \=(*)\\
        \:\psi(x,\!\top)                                \=\mathsf{subs}\\
        \:\alpha                       \=\elim\leftrightarrow(\ref{1})\\
      \*\:\alpha\\
        \:\psi(x,\!\top)               \=\elim\leftrightarrow(\ref{1})\\
        \:\beta=\top                                     \=\mathsf{func}\\
        \:\beta\=(*)\\
      \)\:\alpha=\beta                          \=\intro\leftrightarrow\\
        \:\psi(x,\alpha)                                 \=\mathsf{subs}\\
    \]  \:\psi(x,\alpha)                                  \=\elim\exists\\
    \*  \:\psi(x,\alpha)\\
      \(\:\alpha\\
        \:\alpha=\top                                   \=(*)\\
        \:\phi(x,\!\top)                                \=\mathsf{subs}\\
      \*\:\psi(x,\!\top)\\
        \:\alpha=\top                                   \=\mathsf{func}\\
```

```
      \:\alpha                                        \=(*)\\
    \)\:\alpha\leftrightarrow\psi(x,\!\top)   \=\intro\leftrightarrow\\
  \)  \:\psi(x,\alpha)\leftrightarrow
        (\alpha\leftrightarrow\psi(x,\!\top)) \=\intro\leftrightarrow\\
\end{proofbox}
```

Syntax as follows: each line is of the form

⟨variables⟩ ⟨name⟩ `\:` ⟨formula⟩ `\=` ⟨reason⟩ `\-` ⟨use⟩ `\\`

where

- ⟨variables⟩ is something like "x, y" — it's for variables declared at the beginning of $\forall\mathcal{I}$- and $\exists\mathcal{E}$-boxes.
- ⟨name⟩ is a command `\label{fred}` which defines `fred` to be the label text, which may be used anywhere as `\ref{fred}` — see *The LaTeXbook*. Local labels are also available, using `\lbl{`⟨name⟩`}` or `\"`⟨name⟩`"`; these obey the scoping rules of the boxes. You may also refer to the previous line as `\ref{-}`.
- ⟨formula⟩ is the proposition being asserted.
- ⟨reason⟩ is `\intro\land(\ref{john},\ref{mary})` or `\elim\forall(\ref{jim})`.
- ⟨use⟩ is provided for linear logic, to record the step which uses this one. How this accords with theory I don't yet know.

Note that the parts are separated by `\:`, `\=` and `\\`; these correspond to

let ⟨name⟩ = ⟨expression⟩ : ⟨type⟩

in a declarative language. The `\:`, `\=` and `\-` fields are optional and may occur in any order. If any of them is repeated the last is taken. If none of them is present the ⟨variables⟩ field is also ignored.

Proof *boxes* are "wrapped up" as follows:

- the whole proof in `\begin{proofbox}`...`\end{proofbox}`;
- single-column boxes ($\forall\mathcal{I}$, $\rightarrow\mathcal{I}$, $\exists\mathcal{E}$), in `\[`...`\]`.
- multiple-column boxes are of two kinds:
  - separate ($\land\mathcal{I}$) boxes: `\(`...`\*`...`\)`.
  - stuck together ($\lor\mathcal{E}$) boxes: `\(`...`\+`...`\)`.

You may put more than two columns in `\(`...`\)` and even mix the `\+` and `\*` separators.

The whole proof is enclosed in `\proofbox`...`\endproofbox` or `\begin{proofbox}`...`\end{proofbox}`, but the LaTeX environment form *must not* be used for nested boxes.

If the proof occurs in paragraph mode (ie in vertical or unrestricted horizontal mode) then it is set as a display, using the full width of the page. Otherwise it uses only the required width.

A lot of the internals are potentially configurable, but there is not yet a user interface suitable for doing this. This will be provided in the next version.

2 Redefinable macros

WARNING: most of these commands will be hidden and replaced with optional arguments to `\proofbox` in a future version. Do not rely on them.

We provide three different ways of numbering the lines of the proof:

- `\runningproofline`: a global running sequence (default),
- `\nestedproofline`: a hierarchical system with dots,
- `\nestedproofline`: a fully hierarchical system which also includes the column number (`\proof@columns`) as a letter (ASCII quote plus number).

`\theproofline` is the default.

The macro `\proofboxmakelabel#1` is used to print the line label. We only put it in the leftmost box. It is printed in small non-ranging Arabic numerals (0123456789). Right-justify it in `\prooflinenowidth` if it will fit, otherwise let it stick out on the right, *i.e.* left-justify it.

Kill the numbers altogether with `\proofboxnonumbers`.

How to make the left column of the proof box: use the variables field, a space if necessary and the line label.

How to make the middle column of the proof box: left justify the formula field.

How to make the right column of the proof box: use the reason and use fields.

Make the four edges of a rectangular box and the separator between `\+` columns.

Use dotted lines: `\dottedproofbox`.

Leave the boxes open at the bottom: `\openproofbox`

3 Miscellaneous logical notations

These macros are now in my `logicsym.sty`

Print the names of the introduction and elimination rules, for example:

`\elim\forall` $\forall\mathcal{E}$ `\intro\land` $\land\mathcal{I}$

Recall that in TeX the logical connectives and quantifiers are called

`\lor` $\lor$ `\land` $\land$ `\lnot` $\lnot$ `\forall` $\forall$ `\exists` $\exists$

The following provide macros for the `\implies` *relation* and for the binary *operation* which yields the abstract `\implic`ation between formulae. The point is that TeX spaces them and breaks the lines differently:

`A\implies B` $A \to B$ *versus* `A\implic B` $A{\to}B$

There are forward and reverse, single and Double versions.

Handle the spacing after a variable (and optionally its type) bound by a quantifier symbol. For example

`\All x:X. \phi(x)` prints as $\forall x{:}X.\ \phi(x)$ instead of $\forall x : X.\phi(x)$

We provide some commonly used forms; `\iota` (ι) is Russell's description operator and should really be inverted. There are several notations for substitution. After writing $a[x := b]$ throughout my book I thought I might change to $[b/x]^*a$. This macro reads the source in the first form and prints in the second. If you use it you can, like me, defer the decision about which notation to use until the final stages, doing

```
\renewcommand{\Subst}{\plainsubstitution}
```

if you finally decide on making substitution act on the right. This is already an improvement on the literal text, because it automatically enlarges the brackets according to the text inside. `\Subst` itself is (following my book) defined in terms of the action of a context morphism (`\CtxtMor`) on a term. Again you can do

```
\renewcommand{\CtxtMor}{\plaincontextmorphism}
```

for something simpler. This macro interprets its argument as a comma-separated list of items in the form $x := b$, which it switches to b/x. The simple versions.

4 Some very easy logic exercises

The following examples are taken from Krysia Broda's *Solutions to Problems 5* (KB-Logic-B1-90) and took me a little under an hour to type in.

page 1: (a)

1	$P \wedge Q$	
2	P	$\wedge\mathcal{E}$

1	P	
2	Q	
3	P	(1)
4	$Q \to P$	$\to\mathcal{I}$

page 2: (c)

1	P	
2	Q	
3	$P \wedge Q$	$\wedge\mathcal{I}(1,2)$
4	$Q \to (P \wedge Q)$	$\to\mathcal{I}$

(g)

1	$P \to (Q \to R)$	
2	$P \to Q$	
3	P	
4	Q	$\to\mathcal{E}(2,3)$
5	$Q \to R$	$\to\mathcal{E}(1,3)$
6	R	$\to\mathcal{E}(5,4)$
7	$P \to R$	$\to\mathcal{I}$
8	$(P \to Q) \to (P \to R)$	$\to\mathcal{I}$

page 3: (h)

1	$P \to (Q \to R)$	
2	$P \wedge Q$	
3	P	$\wedge\mathcal{E}1(2)$
4	$Q \to R$	$\to\mathcal{E}(1,3)$
5	Q	$\wedge\mathcal{E}2(2)$
6	R	$\to\mathcal{E}(4,5)$
7	$P \wedge Q \to R$	$\to\mathcal{I}$

(i)

1	$P \wedge Q \to R$	
2	P	
3	Q	
4	$P \wedge Q$	$\wedge\mathcal{I}(2,3)$
5	R	$\to\mathcal{E}(1,4)$
6	$Q \to R$	$\to\mathcal{I}$
7	$P \to (Q \to R)$	$\to\mathcal{I}$

(j)

1	$P \to Q$	
2	$\neg Q$	
3	P	
4	Q	$\to\mathcal{E}(1,3)$
5	$\perp$	$\neg\mathcal{E}(2,4)$
6	$\neg P$	$\neg\mathcal{I}$

page 5: (k)

1	$\neg P$	
2	P	
3	$\neg Q$	
4	$\neg P \wedge P$	$\wedge\mathcal{I}(1,2)$
5	$\perp$	$\neg\mathcal{E}(1,2)$
6	$\neg\neg Q$	$\neg\mathcal{I}$
7	Q	$\neg\neg$
8	$P \to Q$	$\to\mathcal{I}$

page 8: (o)

1	$P \to Q$	
2	$\neg Q$	
3	P	
4	Q	$\to\mathcal{E}(1,3)$
5	$\bot$	$\neg\mathcal{E}(2,4)$
6	$\neg P$	$\neg\mathcal{I}$
7	$\neg Q \to \neg P$	$\to\mathcal{I}$

(p) The $\neg\neg$ rule is unnecessary!

1	$P \to Q$	
2	$\neg\neg P$	
3	P	$\neg\neg$
4	$\neg Q$	
5	Q	$\to\mathcal{E}(1,3)$
6	$Q \wedge \neg Q$	$\wedge\mathcal{I}(5,4)$
7	$\bot$	$\neg\mathcal{E}$
8	$\neg\neg Q$	$\neg\mathcal{I}$
9	$\neg\neg P \to \neg\neg Q$	$\to\mathcal{I}$

5 Some more exercises

1	$P \vee \neg P$			
2	P		$\neg P$	
3	$Q \to P$	by (e)	$P \to Q$	by (k)
4	$(P \to Q) \vee (Q \to P)$	$\vee\mathcal{I}$	$(P \to Q) \vee (P \to Q)$	$\vee\mathcal{I}$
5	$(P \to Q) \vee (P \to Q)$			$\vee\mathcal{E}$

1	$P \vee Q$			
2	$(\neg P) \wedge (\neg Q)$			
3	P		Q	
4	$\neg P$	$\wedge\mathcal{E}1(2)$	$\neg Q$	$\wedge\mathcal{E}2(2)$
5	$\bot$	$\neg\mathcal{E}(4,3)$	$\bot$	$\neg\mathcal{E}(3,4)$
6	$\bot$			$\vee\mathcal{E}(1)$
7	$\neg((\neg P) \wedge (\neg Q))$			$\neg\mathcal{I}$

x	1	$\forall x'.x' < x \to p(x')$							
	2	$x = a$				$x = b$		$x = c$	
	3	$a = c$		$c < a$		$a < x$		$b < x$	
	4	$b < x$	subst(**??**)	$c < x$	subst(2)	$p(a)$	$\forall\mathcal{E}$	$p(b)$	$\forall\mathcal{E}$
	5	$p(b)$	$\forall\mathcal{E}(1)$	$p(c)$	$\forall\mathcal{E}(1)$	$a \neq a$		$b \neq b$	
	6	$b \neq b$	$\wedge\mathcal{E}$	$c \neq c$	$\wedge\mathcal{E}$	$\bot$		$\bot$	
	7	$\bot$	refl	$\bot$		$x \neq b$	$\neg\mathcal{I}$	$x \neq c$	$\neg\mathcal{I}$
	8	$\bot$			$\vee\mathcal{E}$(**??**)				
	9	$x \neq a \wedge x \neq b \wedge x \neq c$							$\wedge\mathcal{I}$
	10	$p(x)$							def

	1	$a < b$								
	2									$b < c$
	3	$a < c \vee (a = c \vee c < a)$								
	4	$a = c \vee c < a$								$a < c$
	5	$p(x) \equiv (x \neq a \wedge (x \neq b \wedge x \neq c))$							def	
x	6	$\forall x'.x' < x \to p(x')$								
	7	$x = a$				$x = b$		$x = c$		
	8	$a = c$		$c < a$		$a < x$		$b < x$		
	9	$b < x$	subst(2)	$c < x$	subst(7)	$p(a)$	$\forall\mathcal{E}$	$p(b)$	$\forall\mathcal{E}$	
	10	$p(b)$	$\forall\mathcal{E}(6)$	$p(c)$	$\forall\mathcal{E}(6)$	$a \neq a$		$b \neq b$		
	11	$b \neq b$	$\wedge\mathcal{E}$	$c \neq c$	$\wedge\mathcal{E}$	$\bot$		$\bot$		
	12	$\bot$	refl	$\bot$		$x \neq b$	$\neg\mathcal{I}$	$x \neq c$	$\neg\mathcal{I}$	
	13	$\bot$			$\vee\mathcal{E}(4)$					
	14	$x \neq a \wedge x \neq b \wedge x \neq c$							$\wedge\mathcal{I}$	
	15	$p(x)$							def	
	16	$\forall x.(\forall x'.x' < x \to p(x')) \to p(x)$							$\forall\mathcal{I}$	
	17	$p(a)$							$\forall\mathcal{E}$	
	18	$a \neq a$							$\wedge\mathcal{E}$	
	19	$\bot$								
	20	$a < c$							$\bot\mathcal{E}$	
	21	$a < c$								$\vee\mathcal{E}$

6 Krysia Broda's dragons exercise

	1	$\forall x.\mathsf{happy}(x) \Leftarrow [\forall y.\mathsf{child}(y,x) \Rightarrow \mathsf{fly}(y)] \wedge \mathsf{dragon}(x)$			
	2	$\forall x.\mathsf{green}(x) \wedge \mathsf{dragon}(x) \Rightarrow \mathsf{fly}(x)$			
	3	$\forall x.[\exists y.\mathsf{parent}(y,x) \wedge \mathsf{green}(y)] \Rightarrow \mathsf{green}(x)$			
	4	$\forall z.\forall x.\mathsf{child}(x,z) \wedge \mathsf{dragon}(z) \Rightarrow \mathsf{dragon}(x)$			
	5	$\forall x.\forall y.\mathsf{parent}(x,y) \Leftarrow \mathsf{child}(y,x)$			
$\forall x_0$	6	$\mathsf{dragon}(x_0)$			
	7	$\mathsf{green}(x_0)$			
$\forall y_0$	8	$\mathsf{child}(y_0,x_0)$			
	9	$\mathsf{parent}(x_0,y_0)$	$\forall\mathcal{E}(5)$	$\mathsf{dragon}(y_0)$	$\forall\mathcal{E}(4, x := y_0, z := x_0, 6)$
	10	$\mathsf{parent}(x_0,y_0) \wedge \mathsf{green}(x_0)$	$\wedge\mathcal{I}(7)$		
	11	$\exists z.\mathsf{parent}(z,y_0) \wedge \mathsf{green}(z)$	$\exists\mathcal{I}(z := x_0)$		
	12	$\mathsf{green}(y_0)$	$\forall\mathcal{E}(3, x := y_0)$		
	13	$\mathsf{green}(y_0) \wedge \mathsf{dragon}(y_0)$			$\wedge\mathcal{I}$
	14	$\mathsf{fly}(y_0)$			$\forall\mathcal{E}(2, x := y_0, 13)$
	15	$\forall y.\mathsf{child}(y,x_0) \Rightarrow \mathsf{fly}(y)$			$\forall\mathcal{I}$
	16	$\mathsf{happy}(x)$			$\forall\mathcal{E}(1, 15, 6)$
	17	$\forall x.\mathsf{happy}(x) \Leftarrow \mathsf{green}(x) \Leftarrow \mathsf{dragon}(x)$			$\forall\mathcal{I}$

Where the previous deduction is a premise of a rule, the reference is omitted. Where the substitution is of the same letter (possibly with a subscript) it is omitted, except for the α-conversion (change of bound variable name) in line **??**.

7 Proof boxes from my book

$\forall x :$	1	$\forall x'.x' \prec x \Rightarrow \phi(x')$	induction hypothesis
	2	$\vdots$	
	3	$u \prec x$	various terms u
	4	$\phi(u)$	$\forall\mathcal{E}(1,3)$
	5	$\vdots$	
	6	$\phi(x)$	the property
	7	$\forall x.(\forall x'.x' \prec x \Rightarrow \phi(x')) \Rightarrow \phi(x)$	$\forall\mathcal{I}$
	8	$\forall x.\phi(x)$	$\prec$-induction for ϕ

1	$\phi(0)$	z
2	$\forall n.\phi(n) \Rightarrow \phi(n+1)$	s
3	$\phi(0) \Rightarrow \phi(1)$	$\forall\mathcal{E}(2)$
4	$\phi(1)$	$\Rightarrow\mathcal{E}(3,1)$
5	$\phi(1) \Rightarrow \phi(2)$	$\forall\mathcal{E}(2)$
6	$\phi(2)$	$\Rightarrow\mathcal{E}(5,4)$
7	$\phi(2) \Rightarrow \phi(3)$	$\forall\mathcal{E}(2)$
8	$\phi(3)$	$\Rightarrow\mathcal{E}(7,6)$

	1	$\forall x.(\forall x'.x' < x \Rightarrow \phi(x')) \Rightarrow \phi(x)$	hypothesis
	2	$\psi(y) = \forall x.(fx = y) \Rightarrow \phi(x)$	definition
$\forall y$:	3	$\forall y'.y' \prec y \Rightarrow \psi(y')$	
$\forall x$:	4	$fx = y$	
$\forall x'$:	5	$x' < x$	
	6	$fx' \prec y$	monotonicity
	7	$\psi(fx')$	$\forall\mathcal{E}(3)$
	8	$\phi(x')$	$\forall\mathcal{E}(\text{def } 2, 4)$
	9	$\forall x'.x' < x \Rightarrow \phi(x')$	$\forall\mathcal{I}$
	10	$\phi(x)$	$\forall\mathcal{E}(1)$
	11	$\forall x.(fx = y) \Rightarrow \phi(x)$	$\forall\mathcal{I}$
	12	$\psi(y)$	def(2)
	13	$\forall y.(\forall y'.y' \prec y \Rightarrow \psi(y')) \Rightarrow \psi(y)$	$\forall\mathcal{I}$
	14	$\forall y.\psi(y)$	$(Y, \prec)$-induction

	1	$\forall y.[\forall y'.y' \epsilon\, y \Rightarrow \phi(y')] \Rightarrow \phi(y)$	
$\forall x$:	2	$\forall x'.x' \prec x \Rightarrow \phi(fx')$	
$\forall y'$:	3	$y' \epsilon\, fx$	
	4	$\exists x'.x' \prec x \wedge y' = fx'$	surj on pred
$\exists x'$:	5	$x' \prec x$	
	6	$y' = fx'$	
	7	$\phi(fx')$	$\forall\mathcal{E}(2)$
	8	$\phi(y')$	substitution
	9	$\phi(y')$	$\exists\mathcal{E}(4)$
	10	$\forall y'.y' \epsilon\, fx \Rightarrow \phi(y')$	$\forall\mathcal{I}$
	11	$\phi(fx)$	$\forall\mathcal{E}(1, y := fx)$
	12	$\forall x.[\forall x'.x' \prec x \Rightarrow \phi(fx')] \Rightarrow \phi(fx)$	$\forall\mathcal{I}$

	1	$\forall x_2, x.[x_2 \ll x \leftrightarrow x_2 \prec x \vee \exists x_1.x_2 \ll x_1 \prec x]$	
	2	$\forall x.[\forall x'.x' \prec x \Rightarrow \phi(x')] \Rightarrow \phi(x)$	hypothesis
	3	$\psi(x) = \forall x_2.x_2 \ll x \Rightarrow \phi(x_2)$	definition
	4	$\forall x.\psi(x) \Rightarrow \phi(x)$	$\forall\mathcal{I}(\forall\mathcal{E}(2,3))$
	5	$\phi(x)$	
	6	$\forall x_2.x_2 \ll x \Rightarrow \phi(x_2)$	
$\forall x_1:$	7	$x_2 \ll x_1$	
	8	$x_2 \ll x$	
	9	$\phi(x_2)$	
	10	$\forall x_2.x_2 \ll x \Rightarrow \phi(x_2)$	
	11	$\psi(x_1)$	
	12	$\forall x_1.x_1 \prec x \Rightarrow \psi(x_1)$	

$\forall x:$	$\forall x_1.x_1 \prec x \Rightarrow \psi(x_1)$				
$\forall x_2:$	$x_2 \ll x$				
	$x_2 \prec x_1 \vee \exists x_1.x_2 \ll x_1 \prec x$				$\forall\mathcal{E}(1)$
	$x_2 \prec x_1$		$\exists x_1:$	$x_2 \ll x_1 \prec x$	
	$\psi(x_2)$	$\forall\mathcal{E}(5)$		$\psi(x_1)$	$\forall\mathcal{E}(5)$
	$\phi(x_2)$	$\forall\mathcal{E}(4)$		$\phi(x_2)$	$\forall\mathcal{E}(3, x := x_1)$
	$\phi(x_2)$				$\exists\vee\mathcal{E}(7)$
	$\forall x_2.x_2 \ll x \Rightarrow \phi(x_2)$				$\forall\mathcal{I}$
	$\psi(x)$				def(3)

	14	$\forall x.[\forall x_1.x_1 \prec x \Rightarrow \psi(x_1)] \leftrightarrow \psi(x)$	$\forall{\leftrightarrow}\mathcal{I}$

	1	$\forall U.[\forall V.V \prec^\flat U \Rightarrow \phi(V)] \leftrightarrow \phi(U)$	
	2	$\phi(\emptyset)$	
$\forall x, U:$	3	$\phi(U)$	
	4	$[\forall y.y \prec x \Rightarrow [\forall V.\phi(V) \Rightarrow \phi(V \cup \{y\})]]$	
$\forall V_0:$	5	$V_0 \prec^\flat U$	
	6	$\phi(V_0)$	premise
	7	$\emptyset \prec^\flat \{x\} \Rightarrow \phi(V_0)$	
$\forall y, V_1:$	8	$V_1 \prec^\flat x$	
	9	$\phi(V_0 \cup V_1)$	
	10	$y \prec x$	
	11	$\phi(V_0 \cup V_1 \cup \{y\})$	$\forall\mathcal{E}(4, 10, V := V_0 \cup V_1, 9)$
	12	$\forall y, V_1.V_1 \prec^\flat x \wedge y \prec x \wedge \phi(V_0 \cup V_1) \Rightarrow \phi(V_0 \cup V_1 \cup \{y\})$	$\forall\mathcal{I}$
	13	$\forall V_1.V_1 \prec^\flat \{x\} \Rightarrow \phi(V_0 \cup V_1)$	K-induction
	14	$\forall V_0, V_1.V_0 \prec^\flat U \wedge V_1 \prec^\flat \{x\} \Rightarrow \phi(V_0 \cup V_1)$	$\forall\mathcal{I}$
	15	$\forall V.V \prec^\flat U \cup \{x\} \Rightarrow \phi(V)$	Lemma
	16	$\phi(U \cup \{x\})$	premise
	17	$\forall x.[\forall y.y \prec x \Rightarrow [\forall V.\phi(V) \Rightarrow \phi(V \cup \{y\})]] \Rightarrow [\forall U.\phi(U) \Rightarrow \phi(U \cup \{x\})]$	$\forall\mathcal{I}$
	18	$\forall x.[\forall U.\phi(U) \Rightarrow \phi(U \cup \{x\})]$	$\prec$-induction
	19	$\forall U.\phi(U)$	K-induction

ϕ	
$\vdots$	
ψ	
$\phi \Rightarrow \psi$	$\Rightarrow\mathcal{I}$

$\vdots$	
$\phi(x')$	
$\forall x.\phi(x)$	$\forall\mathcal{I}$

$\phi \vee \psi$

ϕ	ψ
$\vdots$	$\vdots$
α	α

α $\quad\vee\mathcal{E}$

$\exists x.\phi(x)$

$\exists x'$:	$\phi(x')$
	$\vdots$
	α

α $\quad\exists\mathcal{E}$

	1	$\alpha \leftrightarrow \psi(x, \top)$	
	2	$\exists\beta.\psi(x,\beta)$	total
$\exists\beta$:	3	$\psi(x,\beta)$	
	4	β	
	5	$\beta = \top$	
	6	$\psi(x,\top)$	subs
	7	α	$\leftrightarrow\mathcal{E}$
	8	$\alpha = \beta$	$\leftrightarrow\mathcal{I}$
	9	$\psi(x,\alpha)$	subs
	10	$\psi(x,\alpha)$	$\exists\mathcal{E}$

α	
$\psi(x,\top)$	$\leftrightarrow\mathcal{E}$
$\beta = \top$	func
β	

$\psi(x,\alpha)$	
α	$\psi(x,\top)$
$\alpha = \top$	$\alpha = \top$ func
$\phi(x,\top)$ subs	α
$\alpha \leftrightarrow \psi(x,\top)$	$\leftrightarrow\mathcal{I}$

11 $\psi(x,\alpha) \leftrightarrow (\alpha \leftrightarrow \psi(x,\top))$ $\quad\leftrightarrow\mathcal{I}$

1	α	
2	$\alpha \Rightarrow \alpha$	
3	α	(1)
4	$\alpha \Rightarrow (\alpha \Rightarrow \alpha) \Rightarrow \alpha$	$\forall\mathcal{I}$
5	(0)	

1	α	
2	$\alpha \Rightarrow \alpha$	
3	α	$\Rightarrow\mathcal{E}(2,1)$
4	$\alpha \Rightarrow (\alpha \Rightarrow \alpha) \Rightarrow \alpha$	$\forall\mathcal{I}$
5	(1)	

1	α	
2	$\alpha \Rightarrow \alpha$	
3	α	$\Rightarrow\mathcal{E}(2,1)$
4	α	$\Rightarrow\mathcal{E}(2,3)$
5	$\alpha \Rightarrow (\alpha \Rightarrow \alpha) \Rightarrow \alpha$	$\forall\mathcal{I}$
6	(2)	

	1	$\phi \wedge \psi$	
$\forall\alpha$:	2	$\phi \to (\psi \to \alpha)$	
	3	ϕ	$\wedge 1\mathcal{E}(1)$
	4	$\psi \to \alpha$	$\to\mathcal{E}(2,3)$
	5	ψ	$\wedge 2\mathcal{E}(1)$
	6	α	$\to\mathcal{E}(4,5)$
	7	$\forall\alpha.(\phi \to (\psi \to \alpha)) \to \alpha$	$\forall{\to}\mathcal{I}$

$p_1 : \phi \times \psi$

$p_2 : \phi \to (\psi \to \alpha)$

$p_3 = \pi_1(p_1) : \phi$

$p_4 = p_2(p_3) : \psi \to \alpha$

$p_5 = \pi_2(p_1) : \psi$

$p_6 = p_4(p_5) : \alpha$

$p_7 = \lambda\alpha.\lambda p_2.p_6$

8 From my JSL paper

	1	$\forall U.\,[\forall V.\,V \prec^\flat U \Rightarrow \phi(V)] \leftrightarrow \phi(U)$	
	2	$\phi(\emptyset)$	$\forall\mathcal{E}(1, \mathrm{def}(\prec^\flat))$
	3	$\psi(x) = \forall U.\phi(U) \Rightarrow \phi(U \cup \{x\})$	
$\forall x$	4	$\forall y.y \prec x \Rightarrow \psi(y)$	
$\forall V_0$	5	$\phi(V_0)$	
	6	$\theta(W) = W \prec^\flat \{x\} \Rightarrow \phi(V_0 \cup W)$	
	7	$\theta(\emptyset)$	$\mathrm{def}(6, 5)$
$\forall W$	8	$\theta(W)$	
$\forall y$	9	$W \prec^\flat \{x\} \Rightarrow \phi(V_0 \cup W)$	$\mathrm{def}(6)$
	10	$W \cup \{y\} \prec^\flat \{x\} \equiv W \prec^\flat \{x\} \wedge y \prec x$	
	11	$\phi(V_0 \cup W)$	$\Rightarrow\mathcal{E}(9, 10)$
	12	$\psi(y) \equiv \forall V.\phi(V) \Rightarrow \phi(V \cup \{y\})$	$\forall\mathcal{E}(4, 10), \mathrm{def}(3)$
	13	$\phi(V_0 \cup W \cup \{y\})$	$\forall\mathcal{E}(12, 11)$
	14	$W \cup \{y\} \prec^\flat \{x\} \Rightarrow \phi(V_0 \cup W \cup \{y\})$	$\Rightarrow\mathcal{I}$
	15	$\theta(W \cup \{y\})$	$\mathrm{def}(6)$
	16	$\theta(\emptyset) \wedge \forall y.\forall W.\,\big[\theta(W) \Rightarrow \theta(W \cup \{y\})\big]$	$\wedge\mathcal{I}(7, \forall\mathcal{I})$
	17	$\forall W.\theta(W)$	K-induction for θ
	18	$\forall W.\,W \prec^\flat \{x\} \Rightarrow \phi(V_0 \cup W)$	$\mathrm{def}(6)$
	19	$\forall V_0.\,\phi(V_0) \Rightarrow \big(\forall W.\,W \prec^\flat \{x\} \Rightarrow \phi(V_0 \cup W)\big)$	$\forall\mathcal{I}$
$\forall U$	20	$\phi(U)$	
	21	$U = \emptyset \vee U \neq \emptyset$	Proposition **??**
	22	$U = \emptyset \Rightarrow \forall W.\,W \prec^\flat (U \cup \{x\}) \Rightarrow \phi(W)$	$\forall\mathcal{E}(19, V_0 = \emptyset, 2)$
$\forall V_0$	23	$V_0 \prec^\flat U$	
$\forall W$	24	$W \prec^\flat \{x\}$	
	25	$\phi(V_0)$	$\forall{\Leftarrow}\mathcal{E}(1, 20, 23)$
	26	$\phi(V_0 \cup W)$	$\forall\mathcal{E}(19, 25, 24)$
	27	$\forall V_0, W.\,V_0 \prec^\flat U \wedge W \prec^\flat \{x\} \Rightarrow \phi(V_0 \cup W)$	$\forall\mathcal{I}$
	28	$U \neq \emptyset \Rightarrow \forall V.V \prec^\flat (U \cup \{x\}) \Rightarrow \phi(V)$	Lemma **??**
	29	$\forall V.V \prec^\flat (U \cup \{x\}) \Rightarrow \phi(V)$	$\vee\mathcal{E}(21, 22, 28)$
	30	$\phi(U \cup \{x\})$	$\forall\mathcal{E}(1, 29)$
	31	$\forall U.\phi(U) \Rightarrow \phi(U \cup \{x\}) \equiv \psi(x)$	$\forall\mathcal{I}, \mathrm{def}(3)$
	32	$\forall x.[\forall y.\,y \prec x \Rightarrow \psi(y)] \Rightarrow \psi(x)$	$\forall\mathcal{I}$
	33	$\forall x.\psi(x)$	$\prec$-induction for ψ
	34	$\phi(\emptyset) \wedge \forall x.\,\forall U.\,\big[\phi(U) \Rightarrow \phi(U \cup \{x\})\big]$	$\wedge\mathcal{I}(2, \mathrm{def}(3, 33))$
	35	$\forall U.\,\phi(U)$	K-induction for ϕ

References

[1] K. Broda, S. Eisenbach, H. Khoshnevisan, and Steven Vickers. *Reasoned Programming.* International Series in Computer Science. Prentice Hall, 1994.

[2] Paul Taylor. Intuitionistic sets and ordinals. *Journal of Symbolic Logic*, 61:705–744, 1996.

[3] Paul Taylor. *Practical Foundations of Mathematics.* Number 59 in Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1999.