Well Founded Coalgebras and Recursion

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Abstract

We define well founded coalgebras and prove the recursion theorem for them: that there is a unique coalgebra-to-algebra homomorphism to any algebra for the same functor. The functor must preserve monos, whereas earlier work also required it to preserve their inverse images. The argument is based on von Neumann's recursion theorem for ordinals. Extensional well founded coalgebras are seen as initial segments of the free algebra, even when that does not exist.

The assumptions about the underlying category, originally sets, are examined thoroughly, with a view to ambitious generalisation. In particular, the "monos" used for predicates and extensionality are replaced by a factorisation system. Future work will obtain much more powerful results by using this in a categorical form of Mostowski's construction that imposes extensionality.

Contents:

A lot of this paper has been re-written in the past two years.

Categorical set theory is the study of *ideas* from Set Theory *as ordinary mathematical objects, without their foundational pretensions*. The first application of our subject was to show that the logic of an elementary topos with natural numbers is more or less the same as Zermelo Set Theory [Mik22, Osi74].

In this paper we study well-foundedness, extensionality and the arguments behind von Neumann's recursion theorem and the Mostowski extensional quotient. We strip them of everything else from their set- (or even topos-) theoretic origins and then identify what is needed of some completely different setting for them to be re-deployed there.

The value of these particular ideas is that they provide ways of expressing very strong principles of *induction* and *recursion*. These may be used in Proof Theory to prove the *consistency* of other logical systems and in Process Algebra to investigate termination or persistence of processes and ask whether one process is "the same" as another.

The role of category theory is that it is a good tool for isolating the essential features of a mathematical argument, whilst demanding nothing by way of foundational beliefs.

In the first section we give the traditional ideas from Set Theory and universal algebra that we are seeking to emulate. Chief among these is the theorem of Johann (John) von

Neumann that defines functions by *recursion* over well-founded relations, *i.e.* those for which we have *induction* for predicates.

Section 2 introduces a novel order-theoretic fixed point theorem and the notion of bisimulation that captures set-theoretic inclusion, coalgebra homomorphisms and process algebra. It demonstrates how the new fixed point theorem provides recursive constructions for relations in preparation for doing this with coalgebras.

Section 3 begins our categorical treatment by showing how coalgebras for a functor put these properties of Set Theory and term algebras in a common abstract setting, summarising the earlier work.

Section 4, which you should omit on first reading, examines our precise requirements of the category and its notion of "mono". In future work this will enable a considerable generalisation of similar previous results from **Set** to other categories.

Section 5 shows how well founded coalgebras are generated and Section 6 proves our central result, the recursion theorem.

Section 7 introduces extensional well founded coalgebras and shows how they behave like "transitive sets" in Set Theory.

Section 8 shows how to *impose* well-foundedness and extensionality on a coalgebra, giving adjoints to the inclusions of categories, on the additional assumption of image factorisation.

Section 9 re-introduces the requirement that the functor preserve inverse images and proves the (relatively few) results that depend on this. Finally, Section 10 considers binary joins, in particular the "overlapping" union in Set Theory and concludes with a survey of whether other there could be other adjoints.

1 Background

We are going to study the Axioms of *Foundation* and *Extensionality* from Set Theory.

The axiom of foundation and the notion of a well founded relation are the (to us, obvious) generalisation of the well-*orderings* or *ordinals* $(X, \prec)$ that Georg Cantor introduced [Can83, Can95, Can97]. He stated their defining property in three ways:

(a) every proper subset $U \subsetneq X$ has a $\prec$-least successor (1883);

(b) every non-empty subset $\emptyset \neq V \subset X$ has a $\prec$-least element (1897); or

(c) there is no infinite descending sequence $\cdots \prec d \prec c \prec b \prec a$.

In fact Euclid had invoked the third of these principles for the natural numbers long beforehand, in *Elements* VII 31, *cf.* [Fow94, p 262].

In Cantor's later account [Can97, §13], the notion of *initial segment* (Abschnitt) plays a far more important role than well-ordering itself. We will see the same in this paper and, when our work is generalised to more complex structures, it will be essential to identify their analogue of initial segments.

Dimitry Mirimanoff recognised the weaker concept of well-*foundedness*, using it to show that Ernst Zermelo's Set Theory and infinitary proofs are not vulnerable to circular arguments like Russell's [Rus02] or Burali-Forti's [BF97] Paradoxes [Mir17a, Mir17b, Mir19]. He also introduced the notion of *rank* that we will see later and anticipated the *hierarchy* of sets and representation of ordinals that came to be associated with John von Neumann. His style is in the same spirit as our own, treating membership like any ordinary relation.

If we define well-foundedness using Cantor's later properties (b) or (c), we have to make frequent use of *excluded middle* or *dependent choice*, respectively. His *original* definition (a) is essentially the intuitionistic one, identifying what we actually want to *do* with the notion, but it is difficult to say when this was adopted formally, without qualification, as the intuitionistic definition of well-foundedness.

Definition 1.1 A binary relation $\prec$ on a carrier A is ***well founded*** if it obeys the ***induction scheme***

$$\frac{\forall a\!:\!A.\ \ (\forall b\!:\!A.\ \ b \prec a \Rightarrow \phi b) \Longrightarrow \phi a}{\forall a\!:\!A.\ \ \phi a}$$

for any predicate ϕ on A.

It will be convenient to dissect this triply nested implication. The innermost one,

$$\forall b\!:\!A.\ b \prec a \Longrightarrow \phi b$$

is standardly called the ***induction hypothesis*** (for ϕ at a).

When ordinary mathematicians *use* induction to prove something, their effort goes in to justifying the middle implication (the one with the long arrow, above the line), which *uses* the induction hypothesis to prove the *next* case. However, our focus is on the *validity* of induction, *i.e.* the outermost implication (written *as* the line) that deduces the *general* case. Therefore we call the whole of the top line the ***induction premise*** (for ϕ).

We recognise that the middle implication is typically not two-way (*â priori*, but of course it always *becomes* two-way *after* we have invoked induction), but in the situation where it is we call it ***tight***, the one-way version being ***loose***.

Remark 1.2 This still leaves the variable ϕ free. For simplicity we will regard well-foundedness in this paper as *quantified over all* ϕ. However, the word *scheme* in the name indicates that we may restrict attention to individual predicates or to a class of them of a certain logical complexity, such as those with at most a particular number of alternations of quantifiers. Our categorical structure is in fact able to accommodate this generalisation (Assumption 4.17) and we will indicate for what predicates we are using induction. However, we do not exploit this generality here, and when that is done it may be necessary to re-examine the claims that our results are independent of the choice.

Example 1.3 With the successor relation $n \prec n+1$ on the natural numbers, the induction scheme is known as Peano induction:

$$\frac{\phi 0 \qquad \forall n\!:\!\mathbb{N}.\ \ \phi n \Longrightarrow \phi(n+1)}{\forall n\!:\!\mathbb{N}.\ \ \phi n}$$

although this idiom predates Giuseppe Peano [Pea89] by at least three centuries.

Whilst the abstract notion of well-foundedness is natural and long-established, many mathematicians seem to be reluctant to use it. Instead they say that they are doing induction or recursion on the *length* of a string, the *height* of a tree, its *depth* in computer science, or some other such numerical measure. This is also the way in which iterative or recursive programs are shown to terminate.

The general result that lies behind such usage is this:

Proposition 1.4 If $(A, \prec)$ is well founded and $f : (B, <) \to (A, \prec)$ is ***strictly monotone*** in the sense that

$$\forall b_1 b_2{:}B. \quad b_1 < b_2 \implies fb_1 \prec fb_2$$

then $(B, <)$ is also well founded.

Proof If B has an infinite descending sequence then so does A, which is forbidden. Alternatively, if $\emptyset \neq V \subset B$ then $\emptyset \neq fV \subset A$, so there is a minimal $a \in fV$, where $a = fb$ for some $b \in V$ and this is minimal there. The more difficult intuitionistic proof will be given in Proposition 9.2. □

The distinction between *proving a predicate by induction* and *defining a function by recursion* has been very poorly explained in textbooks, especially before the advent of (functional) programming. Johann (John) von Neumann showed how to derive recursion from induction in the case of the ordinals, in his reformulation of their theory that became the classic one [vN23, footnote] [vN28, § III]. The principal goal of this paper is to see how far we can generalise the setting of this ***well founded recursion theorem***.

This result appears in most Set Theory textbooks (usually without attribution) in a form like the following, where uniqueness is proved *separately*. However, von Neumann originally *included* it as part of the induction predicate and we will see in Section 6 that it is necessary to restore this for our application.

Theorem 1.5 Let $(A, \prec)$ be a carrier with a well founded binary relation and Θ another carrier with a function $\theta : \mathcal{P}\Theta \to \Theta$ that takes an arbitrary subset of Θ as its argument and returns a single element. Then there is a unique function $f : A \to \Theta$ such that

$$\forall a{:}A. \quad fa \; = \; \theta\left(\{fb \mid b \prec a\}\right).$$

We call this equation the ***recursion scheme***, because we do not quantify over (Θ, θ): in this paper we only ever consider a *particular* (though unspecified) target structure.

Proof An ***initial segment*** of A is a subset $B \subset A$ such that

$$\forall bc{:}A. \;\; c \prec b \in B \Longrightarrow c \in B$$

and an ***attempt*** is a partial function $f : A \rightharpoonup \Theta$ whose ***support*** (domain of definition) $B \subset A$ is an initial segment and

$$\forall b{:}A. \;\; b \in B \implies fb \; = \; \theta(\{fc \mid c \prec b\}).$$

(a) There is a unique attempt with empty support.
(b) The union of any directed family of initial segments or attempts is another such.
(c) The restriction of $\prec$ to any initial segment is well founded.
(d) Any two attempts f, g with the same support B are equal, which we prove by induction over $(B, \prec)$ for the predicate

$$\phi b \;\equiv\; (fb = gb).$$

(e) Hence any two attempts with supports B_1 and B_2 agree on $B_1 \cap B_2$ and so may be amalgamated into an attempt with support $B_1 \cup B_2$.

(f) Given any attempt f with support B, there is a successor attempt g with support

$$C \;\equiv\; sB \;\equiv\; \{c : A \mid \forall b{:}A.\;\; b \prec c \Longrightarrow b \in B\} \qquad \text{given by} \qquad gc \;\equiv\; \theta\{b : A \mid b \prec c\},$$

whilst any attempt with support C restricts to B and these constructions are inverse.

(g) In this construction, $C = B$ iff $B = A$, which we prove by induction over $(A, \prec)$ for the predicate

$$\phi a \;\equiv\; (a \in B),$$

indeed "$C = B$ iff $B = A$" is exactly the induction scheme for this predicate .

(h) The required solution to the recursion equation is the union of all of the attempts (a,b,e); this is total because it is fixed by the successor operation (g) and unique by (d). $\square$

It is *essential* to understand the steps of this traditional proof before proceeding with the rest of this paper. We label them because they will each be the subject of lemmas in our categorical proof.

However, we will give our proof in a generality in which Proposition 1.4 *fails* (even though that is plainly an extremely important property of well founded relations). We therefore lose steps (c) and (e) of the proof and so cannot simply form the union of all attempts in the final part.

For these reasons, the next section gives a revised proof of the Theorem, as a guide to the way we subsequently do it categorically.

Remark 1.6 In order to start generalising these ideas, consider first the recursion scheme: θ is the evaluation operation for some sort of *algebra* Θ. In taking a *set* of arguments instead of a *list*, we are saying that θ is *idempotent* and *commutative* with respect to them, but these conditions are inessential.

Indeed, we can consider any ***free theory***, *i.e.* one with no equations at all, but a (possibly infinite) collection Σ of operation symbols, each r of which has a (possibly infinite) ***arity*** $\mathsf{ar}\,(r)$. Then for any set X (of constants, generators, indeterminates or variables as you please), there is a set

$$TX \quad \equiv \quad \coprod_{r:\Sigma} X^{\mathsf{ar}\,(r)}$$

of ***terms*** of depth 1 built from these generators and operation symbols. With no generators, $T\emptyset$ is the set of constants or nullary operation-symbols. Of course TTX is the set of terms of depth 2 and so on.

An ***algebra*** for these operation *symbols* is a carrier Θ that is equipped with an *operation* $\Theta^{\mathsf{ar}\,(r)} \to \Theta$ for each symbol $r : \Sigma$. These may be combined into a single function on the disjoint union:

$$\theta : T\Theta \longrightarrow \Theta.$$

In particular, at least in the case where all of the arities are finite, there is a ***term-*** or ***free algebra*** that is obtained by forming the union A of all of the iterates of T, applied to the empty set. Since we have already done so exhaustively, applying T again to A yields the same thing, so

$$TA \underset{\mathsf{parse}}{\overset{\mathsf{ev}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\cong}}}} A,$$

where ev and parse are the functions that *wrap* some sub-terms in another operation-symbol and *unwrap* them from the outermost one.

Therefore,

$$b \prec a \quad \equiv \quad \exists r.\ (r, b) \in \mathsf{parse}\,(a)$$

defines the ***immediate sub-term relation*** on A. Since A only consists of expressions that are formed by repeated application of the operation symbols, this relation clearly satisfies the "descending sequence" definition of well-foundedness.

Whilst pure mathematicians still typically do induction on the *depth* of such an expression (*cf.* Proposition 1.4), it is increasingly common for theoretical computer scientists and logicians to say directly that this is ***structural induction*** or ***structural recursion*** on the expressions or language instead.

Returning to Set Theory, the second idea that we want to develop is the following — at first sight innocent — property of the $\in$-relation:

Definition 1.7 A (well founded) binary relation $\prec$ such that

$$\forall ab{:}A. \quad (\forall c{:}A.\ \ c \prec a \iff c \prec b) \implies a = b$$

is called ***extensional***. The analogous property of sub-terms in a free algebra is that the parse map is one-to-one, because any term is uniquely determined by its sub-terms (and outermost operation-symbol).

The generalisation from well *ordered* to well *founded* systems introduced "noise" in the form of repetition. Extensionality removes this *so thoroughly* that there are no automorphisms aside from the identity.

Remark 1.8 In this paper we will put the ideas of well-foundedness and extensionality in a more powerful categorical setting. Together they explain many characteristic features of Set Theory — even when we replace full powersets and general subset-selection (which one might suppose to be the most important ingredients) with almost any functor. They are also important properties of term algebras, underlying the algorithm for *unification*, *i.e.* for assigning (sub-)terms to indeterminates in two or more terms so that they match.

In Set Theory, when we form the "union" of two supposedly independent objects, we may find that they already overlap. (Besides being bizarre from the point of view of any other kind of mathematics, this is irritating for those who use set theory as a foundation.) The way that unification "matches up" sub-terms is similar to the overlapping union.

We shall find in Section 7 that the *category* of extensional well founded structures and the appropriate homomorphisms is actually a *pre-order*, *i.e.* there is at most one map between any two objects. When we put two objects together, they (typically) have a non-empty intersection (meet in this order) and therefore an "overlapping" union.

Remark 1.9 Modelling Zermelo's axioms doesn't go any further than this. The singleton operation $\{-\}$ is the unit of the powerset in a monad, giving *un*ordered pairs by binary union and then ordered ones by one of the classic formulae

$$\{\{\{a\}, \emptyset\}, \{\{b\}\}\}, \qquad \{\{a, 1\}, \{b, 2\}\} \qquad \text{or} \qquad \{\{a\}, \{a, b\}\}.$$

However, our categorical theory can be also used to describe the ordinals, where the monad unit is the successor [Tay25c], *cf.* [JM95, Appx A]. Then binary union gives unordered pairs, but for the classical ordinals these are just maxima and these formulae are no longer useful.

Remark 1.10 Applying universal algebra back to Set Theory, when we take (the functor) T to be the (covariant) powerset $\mathcal{P}$, we see that the terms of successive depth are just sets ($\in$-structures). We often like to have *free algebras* for structures, which in this case would be the ***universal set***, but this does not exist as a legitimate object.

However, the extensional well founded structures are legitimate *fragments* of the universal set. These are known in Set Theory as ***transitive sets***, by which is meant those X for which

$$y \in x \in X \Longrightarrow y \in X, \qquad \text{but not necessarily} \qquad z \in y \in x \in X \Longrightarrow z \in x.$$

The analogue in algebra is a collection of terms that includes all of their sub-terms. This is a familiar situation: a language processor such as a compiler forms just such a collection when it parses a particular program or text.

Such structures are *parts* of the free algebra, whether the latter exists legitimately or not. More precisely, the (possibly illegitimate) *union* (colimit) of the preorder of extensional well founded structures is the free algebra.

Remark 1.11 Continuing with the fiction of the universal set, let's use it as the target Θ of the recursion theorem. Then, for any well founded relation $(A, \prec)$, we may define

$$fa \qquad \text{recursively as the set ($\in$-structure)} \qquad \{fb \mid b \prec a\}.$$

Even if $(A, \prec)$ was not extensional, the result is, because Θ is extensional by the axioms of Set Theory.

Therefore, following Andrzej Mostowski [Mos49, Thm 3],

(a) any extensional well founded relation is isomorphic to a unique set ($\in$-structure); and
(b) any well founded relation has an extensional quotient, with a suitable universal property.

The first of these obliges us to subscribe to the ontological belief that a set *is* some particular thing, instead of having a mathematical property that is shared by any isomorphic structure. There is the same distinction between von Neumann's formulation of the *ordinals* [vN23] and Cantor's *well ordered* sets. Moreover, if we admit that, then we commit ourselves even more deeply, because this $\in$-structure is not defined within Zermelo's original Set Theory [Zer08b], but requires the Axiom-Scheme of Replacement.

Since we are not using Set Theory as our foundations, we do not need to be concerned with Replacement (as yet). On the other hand, the second statement is an ordinary theorem of higher order logic. It's a quotient, so we may construct it using an equivalence relation. This is done symbolically in Theorem 2.17 and in a more general categorical form in Section 8.

Remark 1.12 This discussion of whether Mostowski's construction requires Replacement or not is a distraction. There undoubtedly are constructions that ordinary mathematicians do, but which are not available in Zermelo Set Theory or its modern substitutes:

(a) It is common to *iterate* constructions, either over $\mathbb{N}$ or an ordinal, the simplest example being $\bigcup \mathcal{P}^n(\mathbb{N})$.
(b) By methods variously known as realisability , gluing or logical relations, one can compare the term model of a logic system with a semantic one to prove consistency or

completeness. Since this seems to conflict with Gödel's Incompleteness Theorems, the recursion over the term model must be one that goes beyond what that logic can prove for itself.

We will make a proposal towards a categorical replacement for replacement in [Tay25a], building on the methods in this paper. One of the applications will be the *characterisation* of transfinite iteration of functors. This will be a new tool in the categorical lexicon, adding an *axiom* to lie alongside, for example, the *definition* of the subobject classifier in a category with finite limits. This defines but does not construct an elementary topos.

Mostowski's construction is nevertheless the conceptual key to this, because our definition of transfinite iteration will be another example of the extensional reflection. However, this is in a framework where we use categorical tools to generalise the notions of "injective" (and "surjective") functions. Sections 4 and 8 explain how this is done.

Remark 1.13 Finally, since we have gone to the trouble of saying how induction and recursion are *schemes*, we should also state our position *vis à vis* two traditions in Set Theory: one that employs *completed infinities* (classes, universes, inaccessible cardinals) and another that eschews them, developing *potential infinities* instead.

Completed infinities feature in ordinary mathematics in the form of *free algebras*, as we have seen. André Joyal and Ieke Moerdijk [JM95], approaching the analysis of Set Theory from this point of view, treated the universes of sets and of (three kinds of intuitionistic) ordinals as the free algebras for the powerset functor together with "successor" functions having various properties.

It was their key contribution to model the small/large or set/class distinction using ideas that had been developed in topology and sheaf theory to handle *open maps*. Their *algebraic set theory* has been developed further by a number of authors [Awo13] and now gives a categorical account of several highly powerful notions in Set Theory.

Type theories also commonly include (multiple) *universes*, because, when the motivations are symbolic formulae, it is quite natural to internalise the whole system within itself. This is also used, for example in Homotopy Type Theory, to provide results that would otherwise be obtained *impredicatively*.

Universes can be expressed in native category theory as *internal* models that are equivalent to full subcategories.

Remark 1.14 Our view, on the other hand, is in the tradition of *potential* infinities. We take on board the fact that we *cannot* solve $X \cong \mathcal{P}(X)$, *i.e.* that there are functors such as the covariant powerset that have *no* free algebra. In place of this, we characterise and work with *fragments* of what ought to be the free algebra. In the case of the powerset, these fragments are the $\in$-structures or transitive sets of traditional Set Theory.

Working without completed infinities is also important if we want to understand Replacement, because of the way that it can be dismissed as apparently trivial in the context of universes. (In fact, that approach requires an axiom called Collection to turn large things into small one.) Somehow Replacement allows us to express *very large* things using *small* specifications, like an architect's *plan* for a *skyscraper*, even without an encompassing universe [Tay25a].

Remark 1.15 In this setting we therefore need to explain what we mean when we write **Set** for the ***category of all*** sets (or whatever) and functors between such categories.

Categorists commonly and happily talk about these without making clear what they mean.

Plainly, to do *sheaf* theory we would need to consider functors $F : \mathcal{X}^{\mathsf{op}} \to \mathbf{Set}$ as legitimate objects, and also collections of them. These are completed infinities, although they can in fact be re-formulated to avoid this by considering fibrations $\mathcal{F} \to \mathcal{X}$ instead.

But we're not going to do sheaf theory in this paper. For us, **Set** and other "large" categories are not really the completed infinities of *all* objects but just a shorthand for the *scheme* that says *what it is to be* an object or morphism of the relevant kind. Similarly, a functor is a *process* that turns an object or morphism of one kind into one of the other, not the completed infinity that collects all instances of this transformation.

2 Induction and bisimulation

In order to prove our *categorical* generalisation of the recursion theorem we need to know about *order-theoretic* fixed points. The heavy classical methods for finding these, such as using transfinite recursion, are discussed in [Tay25b], but here we use a modern *intuitionistic* technique instead. The foundations are still *impredicative* second order predicate calculus, *i.e.* the logic of an elementary topos.

We also recall the properties of simulation and bisimulation that both the set-theoretic membership relation and coalgebra homomorphisms obey. As examples of the new fixed point theorem and its associated induction, we prove the results about well founded *relations* that are analogous to those about well founded *coalgebras* that are the main subject of this paper.

Definition 2.1 Let $(X, \leq)$ be a poset (partially ordered set), so the relation $(\leq)$ is reflexive, transitive and antisymmetric ($x \leq y \leq x \Rightarrow x = y$). Then
(a) a subset $I \subset X$ is ***directed*** if

$$\exists x\!:\!X.\, x \in I \qquad \text{and} \qquad \forall xy \in I.\, \exists z \in I.\, x \leq z \geq y;$$

(b) $(X, \leq)$ is a ***dcpo*** (directed-complete poset) if it has joins of all directed subsets, written $\bigvee^{\uparrow}$ or $\bigcup^{\uparrow}$; and
(c) it is a ***ipo*** (inductive poset) if it also has a least element, written $\bot$ or $\emptyset$.

We will show that any endofunction $s : X \to X$ of an ipo has a least fixed point, assuming only that s is ***monotone*** (*i.e.* that it ***preserves order***, not necessarily directed joins),

$$\forall x, y\!:\!X. \quad x \leq y \quad \Longrightarrow \quad sx \leq sy,$$

and ***inflationary***,

$$\forall x\!:\!X. \quad x \leq sx.$$

The key observation for the constructive proof was made by Dito Pataraia in 1997, but he never published it himself before he died in 2011, aged 48. Classical proofs of the fixed point theorem such as [Kur22] expressed it as a *function* of the base point and when directness was first introduced it was called the *compositional* property [MS22], so it is a disgrace that categorists and domain theorists failed to *compose* the *functions* of which we wanted fixed points.

Lemma 2.2 Any dcpo $(X, \leq)$ has a greatest inflationary monotone endofunction, $t : X \to X$. This is idempotent ($\forall x.\ t(tx) = tx$) and its fixed points are exactly the points that are fixed by *all* inflationary monotone endofunctions.

Proof Consider the poset F of all inflationary monotone endofunctions of X, equipped with the pointwise order,

$$r \leq_F s \quad \equiv \quad \forall x{:}X. \quad rx \leq_X sx.$$

This inherits directed joins from the pointwise values in X. Also, id_X is the least element of F, so it is an ipo.

Now, for any $r, s \in F$, the composites $r\,;s$ and $s\,;r$ both lie above both r and s in the pointwise order on F, because

$$\forall x. \quad x \quad \leq \quad rx,\ sx \quad \leq \quad r(sx),\ s(rx),$$

using both the inflationary and monotone properties.

Hence the whole of F is *directed*.

Since F is also directed-*complete*, it therefore has a greatest element, $t : X \to X$. (This is the one impredicative step that we make in the whole paper.)

Now, for any $s \in F$, the composites $s\,;t$ and $t\,;s$ are in F too, so $s\,;t \geq t \leq t\,;s$ by the previous argument, but also $s\,;t \leq t \geq t\,;s$ since t is the greatest element of F. Hence $s\,;t = t = t\,;s$ and in particular $t = t\,;t$.

Finally, if $x = tx$ then $sx = s(tx) = tx = x$ for any $s \in F$. In particular $x \equiv ty$ satisfies this for any $y \in X$. $\square$

We want to use this to investigate the fixed points of *particular* endofunctions, which requires cutting down the dcpo to which we apply this Lemma. Two ways of doing this are considered in [Tay25b], but here we use a third, that had to be developed in order to prove the categorical results in this paper.

Lemma 2.3 Given a monotone endofunction $s : X \to X$ of an ipo $(X, \leq)$ that also has binary meets ($\wedge$), $w \in X$ is a ***well founded element*** if

$$w \ \leq \ sw \qquad \text{and} \qquad \forall x{:}X.\ (sx \wedge w \ \leq \ x) \implies w \ \leq \ x.$$

The function s restricts to the subset $W \subset X$ of well founded elements and then ***any fixed point is maximal***,

$$\forall w \in W.\, \forall x \in X. \quad (x = sx \ \leq \ w) \implies w \ = \ x. \qquad \square$$

This is the poset reduction of Definition 3.5 (which I had in 1995, but I didn't know what to do with it before Lemma 2.2 was pointed out to me). It captures the usual loose form of Definition 1.1, but simpler ones would do: $w \leq sw$ with

$$\forall x{:}X.\ (sx \leq x) \implies w \leq x \qquad \text{or} \qquad \forall x{:}X.\ (sx = x) \implies w \leq x.$$

It would be *highly productive* to characterise the well founded elements (in whichever sense) of any recursive situation. Notice that we only *required* the weaker property of *maximality*, but the next result says that it then actually *has* the stronger one of being the *greatest*. That it is unique could be *far from obvious* in the given setting.

Theorem 2.4 Let $s : X \to X$ be an inflationary monotone endofunction of an ipo in which any fixed point is maximal. Then
(a) X has a greatest element, which we call $\top$;
(b) $\top$ is the unique fixed point of s;
(c) if $\bot$ satisfies some predicate that is preserved by s and directed joins then this also holds for $\top$.

Proof By Lemma 2.2, let $t : X \to X$ be the greatest inflationary monotone endofunction. Then

$$\forall x{:}X. \quad \bot \;\leq\; x \;\leq\; sx \;\leq\; tx \;=\; s(tx),$$

whence

$$\forall x. \quad t\bot \;=\; s(t\bot) \;\leq\; s(tx) \;=\; tx \;\geq\; x,$$

so the $(\leq)$ is equality by maximality of fixed points. Hence $t\bot$ is the greatest element $(\top)$ and is the unique fixed point.

For the final part, the subset $U \subset X$ defined by the predicate is closed under $\bot$, s and $\bigvee^{\!\!\!\uparrow}$. It therefore satisfies the same properties as X itself, so it contains a fixed point, which must be the same as the one in X. □

We call part (c) of the Theorem ***Pataraia induction***, although it was first exploited in a constructive setting by Martín Escardó [Esc03, Thm 2.2] and was essentially already present in Zermelo's two proofs of the well-ordering theorem [Zer04, Zer08a].

Corollary 2.5 The subset $W \subset X$ of well founded elements of any ipo has a greatest element, which is the unique fixed point of s. □

Now we apply these ideas to the notions of well-foundedness and bisimilarity that we will use in the rest of this paper:

Example 2.6 Let $(A, <)$ be any set with a binary relation. The full powerset $\mathcal{P}A$ is an ipo, on which

$$sX \quad \equiv \quad \{a : A \mid \forall b{:}A.\;\; b < a \Longrightarrow b \in X\}$$

is the successor operation from Theorem 1.5(f). Then X is a well founded element of the lattice $\mathcal{P}(A)$ iff it is an initial segment on which $(<)$ is a well founded relation. Also, any binary relation $(A, <)$ contains a largest well founded initial segment, which we call $WA \subset A$.

Proof It is an initial segment iff $X \subset sX$ and the induction premise for U is $sU \cap X \subset U$, *i.e.*

$$\forall a{:}A.\;\; (\forall b{:}A.\; b < a \Rightarrow b \in U) \;\wedge\; a \in X \Longrightarrow a \in U,$$

which is the premise for induction on $(<)$.

Maximality of fixed points is a *relative* version of part (g) of the original proof: if $B' = sB' \subset B = sB \subset A$ with B well founded then $\forall x \in B.\, x \in B'$ by induction. □

The existence of the largest well founded part of a relation seems to have emerged in the 1950s, but there is a more subtle consequence of these ideas:

Corollary 2.7 Any well founded relation admits Pataraia induction on its ipo of initial segments.

Proof The subtlety is that this induction does not come directly from the *relation*: Example 2.6 wrapped that up in the data for Pataraia's Theorem. It is the induction from *there* that is being invoked here. □

The analogue of this for coalgebras is Theorem 5.12, which is key to the applications to recursion that follow it.

The proof of this pudding is in the eating. To illustrate the use of Pataraia's Theorem and its variants, we now prove the analogous results for relations to those that we later discuss for coalgebras, starting with the Recursion Theorem 1.5.

We could start by characterising the well founded *attempts* directly, in the same way as we have just done with relations. However, this would require proving uniqueness by induction.

Instead we recognise that, since the notion of well founded element is defined in terms of the successor and order-theoretic operations, it is essentially an *algebraic* idea rather than a logical one. It is also in this algebraic spirit to restore uniqueness, as in von Neumann's original version.

So we prove an algebraic *isomorphism* that transfers well-foundedness of *elements* one Pataraia structure to another. This will be crucial in the categorical version.

Theorem 2.8 Let $(A, \prec)$ be a carrier with a well founded binary relation and Θ another carrier with a function $\theta : \mathcal{P}\Theta \to \Theta$. Then there is a unique function $f : A \to \Theta$ such that

$$\forall a{:}A. \quad fa \; = \; \theta(\{fb \mid b \prec a\}).$$

Proof We have seen that the initial segments form an ipo $(\mathsf{Seg}, \subset)$, and similarly the attempts form one called $(\mathsf{Att}, \leq)$, for which part (f) of Theorem 1.5 defines a successor too. These are related by a "support" function

$$\mathsf{supp} : \mathsf{Att} \longrightarrow \mathsf{Seg}$$

that commutes with the successor operations, as well as with $\bot$ and $\bigvee^{\!\!\uparrow}$.

The successor doesn't just *extend* an attempt from one with support B to one on sB, but defines a *bijection* between them. Hence $\Phi(B) \Rightarrow \Phi(sB)$, where $\Phi(B)$ is the predicate that says that "there is a *unique* attempt with support B".

We also have $\Phi(\emptyset)$, whilst Φ is preserved by directed unions, since these are colimits, and we do not need to consider binary unions. By Corollary 2.7, $\Phi(A)$ holds, *i.e.* there is a unique attempt with total support. □

We now turn to *extensional* well founded relations, which were the basis of Cantor's theory of the ordinals and the $\in$-structures of Set Theory. Since our point of view is that sets are partial $\mathcal{P}$-algebras, we adapt the recursion theorem to allow its target (Θ, θ) to be partial. This makes it rather more complicated, so it is correspondingly less obvious that there is a *greatest* attempt. Pataraia's theorem comes to the rescue, showing that the easier test of *maximality* suffices.

For this we need the notion, introduced by Mirimanoff as "isomorphisme" [Mir17a, §2], that spells out what equality of sets means for their elements, but it is nowadays best known in Process Algebra [San11]:

Definition 2.9 A function $f : B \to A$ between sets with binary relations is called a ***simulation*** if it has the "lifting" property

$$\forall a'{:}A.\ \forall b{:}B. \quad a' \prec_A fb \implies \exists b'{:}B.\ a' = fb' \ \wedge\ b' \prec_B b.$$

We may define a similar property for a *relation* $(\sim) : B \rightharpoondown A$ instead of f:

$$\forall a a' {:} A.\ \forall b {:} B. \quad a' \prec_A a \ \land\ b \sim a \implies \exists b' {:} B.\ b' \sim a' \ \land\ b' \prec_B b$$

and then $\sim$ is a ***bisimulation*** if it also satisfies the symmetrical property,

$$\forall a {:} A.\ \forall b\, b' {:} B. \quad b' \prec_A b \sim a \implies \exists a' {:} A.\ b' \sim a' \ \land\ a' \prec_A a.$$

Since this makes the empty relation a bisimulation, we need to be clear whether we are talking about functions or relations. Also, the definition is finitary, so it is closed under directed unions:

Lemma 2.10 The bisimulation relations between any two sets with binary relations form an ipo under inclusion. □

Lemma 2.11 The ***successor*** $b \approx a$ of a bisimulation relation $(\sim) : (B, \prec_B) \rightharpoondown (A, \prec_A)$ is defined by

$$\big(\forall a'.\ a' \prec_A a \Rightarrow \exists b'.\ a' \sim b' \ \land\ b' \prec_B b\big) \ \land\ \big(\forall b'.\ b' \prec_B b \Rightarrow \exists a'.\ a' \sim b' \ \land\ a' \prec_A a\big).$$

It extends $(\sim)$ and is also a bisimulation. If $(A, \prec_A)$ is extensional and $(\sim) : B \rightharpoonup A$ is functional,

$$(b \sim a_1) \ \land\ (b \sim a_2) \implies a_1 = a_2,$$

then the successor $(\approx)$ is functional too.

Proof If $b \approx a_1$, $b \approx a_2$ and $a' \prec a_1$ then $\exists b'.\ b' \sim a' \ \land\ b' \prec b$, so

$$\exists b' a''.\ b' \sim a' \prec a_1 \ \land\ b' \sim a'' \prec a_2 \ \land\ b' \prec b,$$

in which $a' = a''$ since $(\sim)$ is functional, so $a' \prec a_2$. The converse is similar, so $a_1 = a_2$ by extensionality of A. □

Proposition 2.12 Between any two well founded relations $(B, \prec_B)$ and $(A, \prec_A)$ there is a greatest bisimulation relation. If A is extensional then the bisimulation is functional and if they are both extensional then it is a partial bijection.

Proof We verify maximality of fixed points for Pataraia induction. So suppose that

$$\forall ab. \quad a \backsim b \iff a \mathrel{\backsimeq} b \implies a \sim b$$

where $\sim$ and $\backsim$ are different symbols, but we will show by a double well founded induction that they are equivalent. The induction hypothesis for this is

$$\forall a' b'. \quad a' \prec_A a \ \land\ b' \prec_B b \ \land\ a' \sim b' \implies a' \backsim b'$$

and then

$$\begin{aligned} a \approx b \ &\equiv\ \big(\forall a'.\ a' \prec_A a \Rightarrow \exists b'.\ a' \sim b' \ \land\ b' \prec_B b\big) \ \land\ \big(\cdots\big) \\ &\Rightarrow\ \big(\forall a'.\ a' \prec_A a \Rightarrow \exists b'.\ a' \backsim b' \ \land\ b' \prec_B b\big) \ \land\ \big(\cdots\big) \quad \equiv \quad a \mathrel{\backsimeq} b, \end{aligned}$$

so

$$a \sim b \implies a \approx b \implies a \mathrel{\backsimeq} b \iff a \backsim b.$$

Thus this holds for all $a \in A$ and $b \in B$.

Hence, using extensionality, the greatest bisimulation is functional by Pataraia induction. When this is so both ways it is a partial bijection. □

Remark 2.13 The well founded induction is another use of Corollary 2.7, applied to initial segments of both orders together, so it operates on the *product* Pataraia structure. The same applies to the use of maximality of fixed points: a bisimulation has supports on both sides that are *initial segments* and its successor relates their successors, as in Theorem 2.8. □

Using Mirimanoff's characterisation of (Set-Theoretic) subset inclusion as a bisimulation, this result says that there is a greatest way of "zipping together" two sets, where the shared part is their set-theoretic intersection. Cantor had done this with the classical ordinals [Can97, §13 Thms N&E], so that one must be an initial segment of the other, since otherwise the least unmatched elements on both sides would extend the matching.

Section 7 re-interprets these ideas for extensional well founded *coalgebras*, which then enjoy very similar properties to Set Theory: All morphisms are mono and they from a preorder with meets. Section 10 then constructs binary unions that are again like those in Set Theory.

Proposition 1.4 said that well-foundedness of relations is reflected by order-preserving functions and in particular is inherited by initial segments, *cf.* Theorem 1.5(c). There is a simpler result about the induction *premise* (*cf.* Definition 1.1), that will be an important tool (Lemma 5.6) in our categorical construction:

Lemma 2.14 Let $f : (B, \prec) \to (A, <)$ be a strictly monotone simulation function and ϕ be a predicate on A that satisfies its induction premise, loosely or tightly. Then substitution preserves this: $\psi b \equiv \phi(fb)$ satisfies the induction premise for B.

Proof Given ϕ and b with

$$\forall a. \quad (\forall a'.\ a' < a \Rightarrow \phi a') \Longrightarrow \phi a \qquad \text{and} \qquad \forall b'.\ b' \prec b \Longrightarrow \psi b',$$

let $a' : A$ be such that $a' < a \equiv fb$. Then, since f is a simulation, there is some lifting $b' : B$ with $a' = fb'$ and $b' \prec b$. By the induction hypothesis for B at b, this satisfies $\psi b'$, which is $\phi(fb')$ or $\phi a'$.

Hence we have proved the induction hypothesis for ϕ on A at $a \equiv fb$. It follows from the induction premise for A that $\phi a \equiv \phi(fb) \equiv \psi b$. Therefore we have proved that

$$\forall b. \quad (\forall b'.\ b' \prec b \Rightarrow \psi b') \Longrightarrow \psi b,$$

which is the induction premise for B.

Strict monotonicity gives the reverse implication for the tight form:

$$b' \prec b \ \wedge\ \psi b \Longrightarrow fb' < fb \ \wedge\ \phi(fb) \Longrightarrow \phi(fb') \equiv \psi b. \qquad \square$$

Corollary 2.15 With the same notation, if f is surjective and B is loosely or tightly well founded then so is A.

Proof If ϕ obeys the induction premise for A in the previous Lemma then $\forall b.\, \psi b$ and $\forall a.\, \exists b.\, a = fb$, whence $\forall a.\, \phi a$ [Tay96a, Lemma 2.7]. □

Now we have further applications of Pataraia induction:

Lemma 2.16 Any simulation function $f : (B, \prec) \to (A, <)$ from an extensional well founded relation to any binary relation whatever is 1–1.

Proof For any initial segment $C \subset B$, let $\Phi(C)$ be the predicate that the composite $C \to B \to A$ is 1–1. This holds for $C \equiv \emptyset$ and is inherited by directed unions.

Suppose $\Phi(C)$ holds and let $b_1, b_2 \in sC$ with $fb_1 = fb_2 \in A$.

Since f is a simulation function, the trivial statement $a < fb_1 \iff a < fb_2$ becomes

$$(\exists b_1'.\, a = fb_1' \;\wedge\; b_1' \prec b_1) \iff (\exists b_2'.\, a = fb_2' \;\wedge\; b_2' \prec b_2),$$

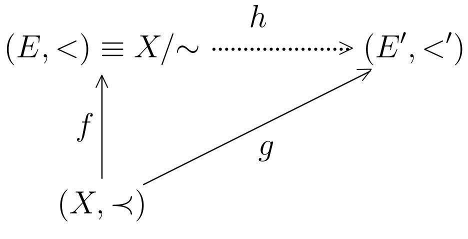
in which we must have $fb_1' = fb_2'$. By construction of sC, we have $b_1', b_2' \in C$, so $b_1' = b_2'$ by $\Phi(C)$. Hence

$$\forall b'.\quad b' \prec b_1 \iff b' \prec b_2,$$

so $b_1 = b_2$ by extensionality of B. So we have proved $\Phi(sC)$. Then Pataraia induction in the form of Corollary 2.7 gives $\Phi(B)$, which is that $f : B \to A$ is 1–1. □

We use these two results for our version of Mostowski's construction (Remark 1.11), which is that any well founded relation may be made extensional by forming the quotient by an equivalence relation. This replaces the *ad hoc* references to co-recursion in [Tay96a, Thm 2.11] with Pataraia induction. Section 8 generalises this to coalgebras.

Theorem 2.17 Let $(X, \prec)$ be a well founded relation. Then there is an extensional well founded relation $(E, <)$ and a surjective simulation function $f : X \to E$, with the universal property that, for any simulation function $g : X \to E'$, where $(E', <')$ is extensional and well founded, there is a unique simulation function $h : E \to E'$ such that $g = h \circ f$.



Proof First consider the universal property. Extensionality of E' at $g(x)$ and $g(y)$, where $x, y \in X$, says

$$\big[\forall e'.\, e' <' g(x) \Leftrightarrow e' <' g(y)\big] \quad\Longrightarrow\quad g(x) = g(y).$$

Write $x \sim y$ for $g(x) = g(y)$ and use Definition 2.9. By an argument similar to Lemma 2.16,

$$(\forall x' \prec x.\, \exists y' \prec y.\, x' \sim y') \quad\wedge\quad (\forall y' \prec y.\, \exists x' \prec x.\, x' \sim y') \quad\Longrightarrow\quad x \sim y,$$

which is a bisimulation relation. By Proposition 2.12, there is a greatest of these and by Pataraia induction it is reflexive, symmetric and transitive (an equivalence relation).

The order relation on the quotient $X/(\sim)$ is defined by

$$[x] \prec [y] \quad \equiv \quad \exists y'.\, x \sim y' \prec y.$$

Then $X \to X/(\sim)$ preserves $\prec$ and is a surjective simulation function. Hence $X/(\sim)$ is well founded by Corollary 2.15.

Since $(\sim)$ is fixed by the successor operation, $X/(\sim)$ is extensional. Moreover, for any denser equivalence relation $(\approx)$, the quotient $X/(\sim) \to X/(\approx)$ is a simulation function out of an extensional well founded relation, so it is 1–1 by Lemma 2.16 and therefore bijective. $\square$

In the next section we will transfer these order- and relation-theoretic ideas to the categorical setting.

3 Well founded coalgebras

We now show how the ideas from Set Theory, universal algebra and process algebra in the previous two sections can be expressed in category theory, building on the work of Christian Mikkelsen and Gerhard Osius. This was done in the years following the introduction of the notion of an elementary topos by Bill Lawvere and Myles Tierney [Law70], when the key issues were to optimise the categorical axioms and show that toposes could do anything that sets could do.

The main part of Mikkelsen's thesis [Mik22] gave an important simplification of the categorical definition of a topos, showing how colimits could be derived from limits, after his supervisor Anders Kock had derived exponentials from powersets [KM74]. As an appendix, he gave the first proof of the recursion theorem in a topos, in a very "diagrammatic" style; apparently he devised the argument himself, not having known von Neumann's Theorem 1.5.

Osius was one of several people who demonstrated how to interpret "ordinary" mathematical notation (higher order logic) in a topos. The aspect of this work that was not also done by other authors was was to take $\in$-structures seriously as mathematical objects in a categorical setting [Osi74, §§4&6]. He also summarised Mikkelsen's proof of the recursion theorem in more familiar notation [Osi75, §6].

It is a pity that neither of them continued studying categorical logic: Osius became a professor of statistics and died in 2019, whilst Mikkelsen became a schoolteacher, having been unable to find a permanent university job.

The extension of their theory to any endofunctor T of a topos that preserves inverse images was made in [Tay99, §6.3] and sketched for other categories in [Tay96b]. In this paper we weaken the requirement on T to preservation of mono(morphism)s, but in the next section we show how the latter may be replaced by special notions of inclusion in other categories.

We give precise references to some corresponding results in these earlier works, for historical comparison, but the ones here are often much more general. (Unfortunately, I mis-attributed Mikkelsen's work to Osius in my earlier work.)

We work throughout in the logic of an elementary topos $\mathcal{S}$, remembering to thank Osius and others for allowing us to write this in the vernacular of mathematics. You may therefore treat $\mathcal{S}$ as **Set**, except that we do not use Excluded Middle or the Axiom of Choice, although the key Lemma 2.2 is impredicative.

So far, we have discussed a binary relation on a carrier A. There are many ways of representing a relation in category or type theory, but the one that we choose is as a function (morphism)

$$A \xrightarrow{\alpha} \mathcal{P}A \qquad \text{by} \qquad a \longmapsto \{b \mid b \prec a\} \subset A,$$

which was already important in the work of Cantor [Can97], Hessenberg [Hes06] and Hartogs [Har15]. It is directly analogous to the parse operation for a free algebra (Remark 1.6), where $\prec$ or $\in$ correspond to the immediate sub-term relation.

We can do the same for any functor T whatever, although we will throughout require it to preserve monos:

Definition 3.1 A ***coalgebra*** for an endofunctor $T : \mathcal{C} \to \mathcal{C}$ of any category is an object A of $\mathcal{C}$ together with a morphism $\alpha : A \longrightarrow TA$. We say (provisionally) that (A, α) is ***extensional*** if α is mono in $\mathcal{C}$, *cf.* Definition 1.7.

$$\begin{array}{ccc} TA & \xrightarrow{Tf} & TB \\ \alpha \uparrow & & \uparrow \beta \\ A & \xrightarrow{f} & B \end{array}$$

A homomorphism of coalgebras is a $\mathcal{C}$-morphism $f : A \longrightarrow B$ that makes the square commute, which we indicate by the triangle arrowhead. We mark the structure map α in the same way because it is a homomorphism to $(TA, T\alpha)$. We write T-**CoAlg** or just **CoAlg** for the category of coalgebras and homomorphisms.

This paper develops an entire theory that is remarkably similar to Set Theory, but just using a functor that need have hardly any of the properties of the powerset. This alone is a massive declaration of foundational autonomy. Nevertheless, to relate coalgebras to the background in Set Theory, we first need a full understanding of the powerset as a functor in a topos:

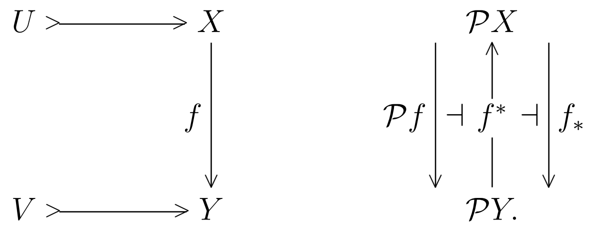
Notation 3.2 The ***covariant powerset functor*** $\mathcal{P} : \mathcal{S} \to \mathcal{S}$ is defined on an object X by $\mathcal{P}X \equiv \Omega^X$ and on a function $f : X \to Y$ by

$$\mathcal{P}fU \;\equiv\; \{fx \mid x \in U\} \;\equiv\; \{y : Y \mid \exists x{:}X.\; y = fx \;\wedge\; x \in U\} \subset Y$$

for $U \subset X$. We shall also need to define, for $V \subset Y$,

$$\begin{array}{rcl} f^*V & \equiv & \{x : X \mid fx \in V\} \\ f_*U & \equiv & \{y : Y \mid \forall x{:}X.\; fx = y \Longrightarrow x \in U\}. \end{array}$$

These also provide the morphism parts of functors $\mathcal{S} \to \mathcal{S}$ that are respectively contravariant and covariant, since $(g\,;f)^*W = f^*(g^*W)$ and $(g\,;f)_*U = g_*(f_*U)$. More importantly for us, there are (order-)adjunctions



Diagrammatically, $\mathcal{P}f$ and f^* are given by composition and pullback respectively. The logical formulae that define $\mathcal{P}fU$ and f_*U are the same except that one involves an existential and the other a universal quantifier. We will use f_* in Section 9. $\square$

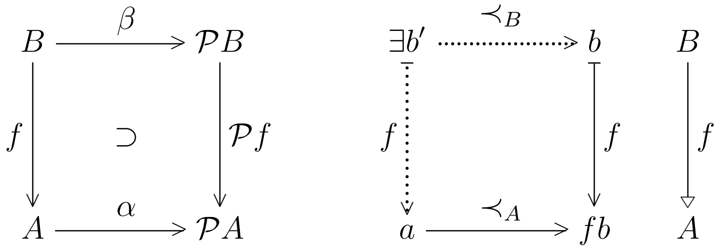
Gerhard Osius's principal insight was to characterise set-theoretic inclusions as homomorphisms of extensional recursive $\mathcal{P}$-coalgebras [Osi74, §6], although we will replace recursion with well-foundedness. The following result combines that with Mirimanoff recognition that subset inclusions are simulations. It appears in [Tay96a, Def 2.3] [Tay96b, Rk 3.4] [Tay99, Rk 6.3.5], but Osius deserves the credit, even though the words coalgebra and simulation had not been introduced: things are only given names *after* they have been observed several times.

Lemma 3.3 A function $f : (B, \prec_B) \to (A, \prec_A)$ is a homomorphism of $\mathcal{P}$-coalgebras iff it is strictly monotone, *i.e.* it preserves the binary relation as in Proposition 1.4,

$$\forall b_1, b_2 : B. \quad b_1 \prec_B b_2 \implies fb_1 \prec_A fb_2,$$

and a simulation (Definition 2.9),

$$\forall a' : A.\ \forall b : B. \quad a' \prec_A fb \implies \exists b' : B.\ a' = fb' \ \wedge\ b' \prec_B b.$$



In this case, the relation $(a \sim b) \equiv (a = fb)$ is actually a *bi*simulation.

Proof The inclusion $\beta\,;\mathcal{P}f \subset f\,;\alpha$ (as marked in the diagram on the left) holds iff f is strictly monotone and the reverse inclusion (illustrated on the right) iff f is a simulation. For a *bi*simulation we also require

$$\forall ab'.\ b' \prec_B b \ \wedge\ fb = a \implies \exists a'.\ b' = fa' \ \wedge\ a' \prec_A a,$$

but this follows from strict monotonicity, with $a' \equiv fb' \prec_A fb = a$. $\square$

Corollary 3.4 If $f : B \subset A$ is a subcoalgebra inclusion then the lifting is unique, so being a homomorphism says that B carries the restriction of $\prec$ from A and is down-closed or an ***initial segment***,

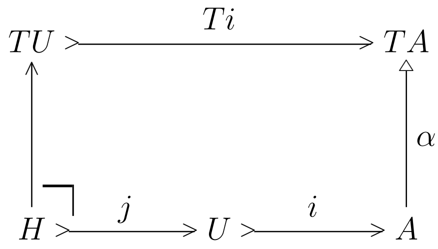
$$\forall ab : A. \quad a \prec b \in B \implies a \in B,$$

just as we have used in the proof of the recursion theorem. $\square$

Observe that (infinitary) *directed* unions of initial segments can only build *ascending* $\prec$-sequences, *cf.* the *descending* sequences that are forbidden by well-foundedness.

We are ready to formulate the two concepts that are connected by our main result.

Definition 3.5 A coalgebra $\alpha : A \longrightarrow\!\!\triangleright\, TA$ is ***well founded*** if in any pullback diagram in the category $\mathcal{C}$ of the form



the maps i and therefore j are necessarily isomorphisms. To clarify, we mean that when we form the pullback H of Ti and α, the map $H \to A$ factors through $i : U \to A$.

Similarly, we say that $\alpha : A \longrightarrow\!\!\triangleright\, TA$ is ***tightly well founded*** if any ordinary pullback (where $H \cong U$) is trivial (so the tight version is *â priori* weaker than the loose one).

We write T-**WfCoAlg** or just **WfCoAlg** for the category of well founded coalgebras and coalgebra homomorphisms. The "scheme" issues in Remark 1.2 will be considered in the next section.

Essentially this "broken pullback" appears (with $T \equiv \mathcal{P}$) on page 67 of [Mik22] and it is written symbolically as $\alpha^{-1}(\mathcal{P}U) \subset U \Longrightarrow U = A$ in [Osi74, §4] and [Osi75, Prop 6.1]. It was first given as the *definition* of well-foundedness in [Tay96b, Tay99].

The result that justifies this name is implicit in the work of Mikkelsen and Osius, but not very clearly expressed there:

Proposition 3.6 A binary relation $(A, \prec)$ is well founded in the earlier sense iff the corresponding (A, α) is a well founded $\mathcal{P}$-coalgebra.

Proof Write $U \equiv \{x : A \mid \phi x\}$ for some predicate ϕ defined on A.

An element $(a, V) \in H \subset A \times TU$ of the pullback consists of $a : A$ and $V \subset U \subset A$ such that

$$\alpha(a) \;\equiv\; \{x : A \mid x \prec a\} \;=\; V.$$

Thus V is determined uniquely by a (and the structure $\alpha : A \longrightarrow\!\!\triangleright\, TA$), but for such a V to exist, a must satisfy

$$\{x : A \mid x \prec a\} \;\subset\; U, \qquad i.e.\;\; \forall x{:}A.\;\; x \prec a \Longrightarrow \phi x.$$

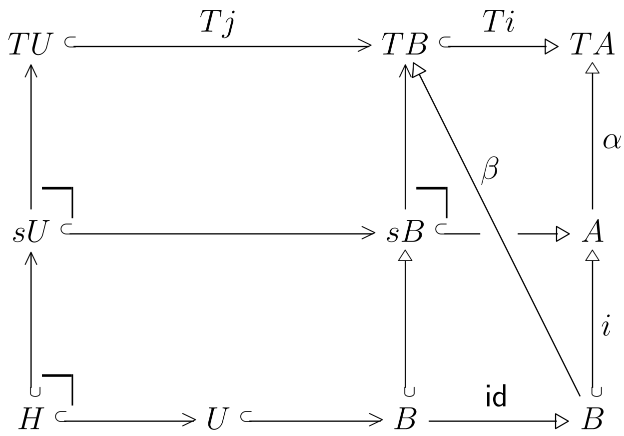
The pullback H therefore corresponds to the induction ***hypothesis*** (Definition 1.1).

The induction ***premise*** is that, for each such $a : A$ that satisfies the hypothesis, we have $a \in U$ or ϕa. In the diagram this means that $H \subset U$. The *tight* induction premise corresponds to having $H \cong U$ instead; this makes $U \subset A$ a subcoalgebra for which the square is a pullback.

Well-foundedness of the coalgebra says that whenever we have a diagram of this form then $U \cong A$, just as the induction *scheme* says that whenever the premise holds then we must have $\forall a{:}A.\; \phi a$. $\square$

We also have agreement with Lemma 2.3, but, since we have promised to discuss well founded *coalgebras*, that is what we will do:

Proposition 3.7 The relative successor can be defined on subobjects of the carrier of a coalgebra. Then such a subobject is a well founded element iff it is a well founded subcoalgebra.

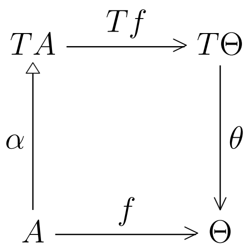


Proof Given a coalgebra $\alpha : A \longrightarrow TA$ and a subobject $i : B \hookrightarrow A$ of its carrier, the (upper right) pullback sB is the successor *subobject*. The condition $B \leq sB$ on subobjects is the lower right square; this is equivalent to having a subcoalgebra structure β, by composition and pullback.

Now let $U \hookrightarrow B$ be a subobject of the carrier B, so its successor sU is defined in the same way. Then the pullback H of TU and B factors through sU and states both
(a) $sU \cap B \equiv H \subset U$ for B to be a well founded *element* and
(b) the broken pullback for (B, β) to be a well founded *coalgebra*.
The tight versions of these are $sU \cap B = U$ and $H \cong U$. □

The other side of the main result is recursion:

Definition 3.8 A *coalgebra* $\alpha : A \longrightarrow TA$ obeys the ***recursion scheme*** if, for every *algebra* $\theta : T\Theta \to \Theta$, there is a unique map $f : A \to \Theta$ such that the square



commutes. The notion is a *scheme* because we only ever consider *particular* algebras (Θ, θ). A map of this kind has also been called a ***coalgebra-to-algebra homomorphism*** [Epp03].

To obtain ***parametric recursion***, in which the top line is replaced by

$$Tf \times \mathsf{id} : TA \times A \longrightarrow T\Theta \times A,$$

we just need to make Lemma 6.5 a bit more complicated. In fact Mikkelsen had an even more general recursion scheme than this, although still with $T \equiv \mathcal{P}$ [Mik22, pp 66–67] [Osi75, Def 6.2]. Osius's account of categorical Set Theory [Osi74] used recursion instead of well-foundedness (induction).

Example 3.9 The predecessor and test for zero function define a coalgebra on $\mathbb{N}$ for the functor $TX \equiv \mathbf{1} + X$ on $\mathcal{S}$. Then recursion defines $f : \mathbb{N} \to \Theta$ by the two cases

$$f0 \;=\; \theta(\star) \qquad \text{and} \qquad fn \;=\; \theta\big(f(n-1)\big).$$

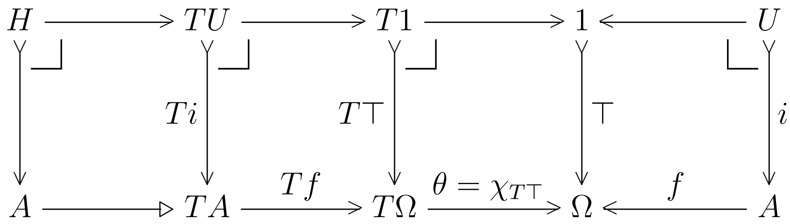
In a topos, well-foundedness is *necessary* for recursion [Mik22, p 68] [Osi75, Prop 6.3] [Tay99, Exercise 6.14]:

Proposition 3.10 In a topos, if $\alpha : A \longrightarrow\!\!\triangleright\, TA$ obeys the recursion scheme then it is well founded.

Proof The subobject classifier (set of truth values) $\Theta \equiv \Omega \equiv \mathcal{P}(1)$ carries an algebra structure for any operation whatever, namely by interpreting it as (infinitary) *conjunction* or universal quantification. Then $f : A \to \Theta$ is a homomorphism iff

$$fa \iff \big(\forall x.\, x \prec a \Rightarrow fx\big).$$

This is the tight ($\Leftrightarrow$) version of the induction premise (Definition 1.1), whilst the constant function $f : a \mapsto \top$ is also a homomorphism. So *uniqueness* of f amounts to the induction scheme.



This argument generalises. Let $\theta : T\Omega \to \Omega$ be the characteristic function of the subset $T\top : T1 \rightarrowtail T\Omega$, where $\top : 1 \rightarrowtail \Omega$ is the element "true". The induction premise is $\alpha\,;Tf\,;\theta \Rightarrow f$ and the tight premise has equality (bi-implication), but this is also satisfied by the constant function with value $\top$. $\square$

Remark 3.11 This result should be treated with circumspection, because taking the object of truth values as the target algebra means that we are relying on *higher order logic*. (This point is obscured classically by the identification of Ω with a discrete two-element set.)

For example, induction for the predicate $\phi x \equiv (x \not\prec x)$ shows that well founded relations must be *irreflexive*. However, this makes the idea too clumsy to analyse *fixed* points of iteration, as we might hope to do in future applications of the theory.

On the other hand, experience shows that we must count ourselves lucky to find a condition for termination of a heavily recursive program which is *sufficient* for the case at hand: asking for it to be *necessary* as well is too much.

Remark 4.18 replaces higher order Ω with similar objects for particular logical complexity levels.

4 Categorical requirements

Our theory applies to an endofunctor T that preserves monos, but we have not yet said anything about what we require of the category $\mathcal{C}$ on which it acts. Beyond that, as we generalise $\mathcal{C}$ further and further away from **Set**, we find that it have many different kinds of "inclusions" that (have but) are not necessarily characterised by the standard cancellation property that defines (what we shall call ***plain***) ***monos*** in a category.

Besides the functor T, the freedom to choose different categories and notions of mono in them gives considerable power to this theory.

We address those questions in this section, but really this is a technical analysis of the proof to follow. Therefore, even if you are proficient in categorical logic, it would still be better to understand the next two or three sections *grosso modo* before reading this one, so that you can see why the following subtleties are needed. So, this is like the configuration section of a piece of software, that logically has to come first, but which you must not touch until you know what you're doing.

On first reading, you should therefore simply take $\mathcal{C} \equiv \mathcal{S} \equiv \mathbf{Set}$, read both arrowtails as injective functions and assume that the functor $T : \mathbf{Set} \to \mathbf{Set}$ preserves them. Then you may omit this section.

The simplest statement of more general but sufficient conditions is this:

Provisional assumption 4.1 The category $\mathcal{C}$
(a) has inverse images (pullbacks) of monos along coalgebra homomorphisms;
(b) has an initial object $\emptyset$ and all maps $\emptyset \to X$ are mono;
(c) has directed unions of subobjects (Definitions 2.1 and 4.3), and
(d) is well powered, whilst
(e) the functor $T : \mathcal{C} \to \mathcal{C}$ preserves monos.

Besides defining "unions" and "well powered", we also need to examine all of these assumptions more carefully.

We will use some other finite limits in $\mathcal{C}$, but only incidentally, not as part of the proof of our main theorem: Lemma 6.6 uses binary products to show how to handle parametric recursion. Lemma 6.4 uses equalisers to prove uniqueness of recursion, but we can deduce that in another way, without using them. The terminal object $\mathbf{1}$ is never used.

Much of this section is about replacing the "monos" in (a,e) with some special class of maps that we use for "predicates" and those in (b,c,d) with another possibly smaller class of "initial segments".

Remark 4.2 Any category of finitary algebras satisfies (a,c,d), but part (b) is more delicate. Recall from universal algebra that, in an appropriately constructed category of algebras, the ***initial object*** typically arises as the collection of terms *generated* by a given set of symbols (*cf.* Remark 1.6).

We can mimic this for any object I of any category: Working instead with the (coslice or cocomma) category whose objects are monos $I \hookrightarrow X$ and whose morphisms are commutative triangles, the initial object is id_I and all maps out of it are monos. This construction leaves the other provisional assumptions intact, because the subobjects, inverse images and directed unions in the coslice are essentially the same as those in the original category.

For example, the category of fields does not meet our requirements as it stands, but cutting it down to *those of a particular characteristic* does: this selects one of the components of the category and then $\mathbb{Q}$ or $\mathbb{F}_{p^n}$ is the initial object. We also need to fix the characteristic if we want to work with rings (or commutative rings), because that ensures that all maps from the initial object ($\mathbb{Z}$ or $\mathbb{Z}_n$) are mono.

Our main Recursion Theorem 1.5 works by building up partial maps from the empty one. This means in particular that the initial *object* must serve as the least *subobject* of any object. This is why maps out of the initial object need to be mono, which is not the case for the initial ring $\mathbb{Z}$.

More generally, in order to *combine* partial maps we need to make the *colimits* of monos in the category behave like *unions* of subobjects of each object, so we first need to be clear what we mean by "unions".

Definition 4.3 A ***union*** in a category is a diagram or its colimit such that
(a) the maps in the diagram are mono;
(b) the maps in the colimiting cocone are mono;
(c) for any other cocone consisting of monos, the colimit mediator is also mono.

This is the same as saying that the inclusion functor $\mathcal{M} \subset \mathcal{C}$ ***creates*** colimits for this diagram, *i.e.* we compute them in $\mathcal{C}$ and then lift the uniquely to $\mathcal{M}$.

Proposition 4.4 Set (or any topos $\mathcal{S}$) has directed unions.

Proof (Sketch) A colimit in **Set** is given by the quotient of a coproduct by an equivalence relation that is obtained from the diagram. The different components of a coproduct are disjoint.

Two elements are identified in the colimiting cocone iff they are linked by a finite zig-zag in the relation. Since the diagram is directed, it has some further stage (beyond the zig-zag but still within the diagram) that is a cocone over the zig-zag. Since this cocone consists of monos, the two elements were already equal.

Now consider the kernel (pullback against itself) of the mediator to any other cocone of monos. Since colimits are stable under pullback, this kernel is a doubly-indexed union. But since the diagram is directed, this is equivalent to a singly-indexed union, which is in fact the original diagram. Hence the projections from the kernel are isomorphisms and so the mediator is mono. □

In other categories, the second part of the argument shows that the mediators in Definition 4.3(c) are *plain* monos whenever colimits are stable under pullback. But this is not sufficient for other kinds of inclusions. We give the analogous results for pushouts in a (pre)topos more formally in Section 10.

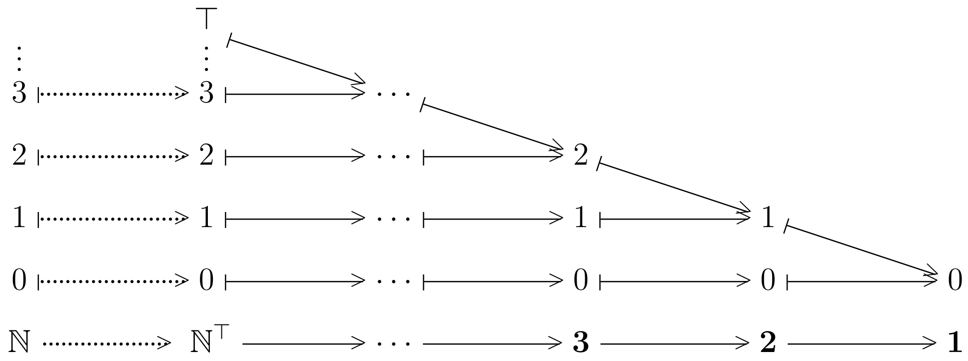
To emphasise the importance of this property, we give an example of its failure:

Example 4.5 The directed union requirement fails for $\mathbf{Set}^{\mathsf{op}}$.

Proof It is clearer to discuss the dual categorical properties in **Set** itself.

Classically, all maps $X \to \mathbf{1}$ are epi, except when $X \equiv \emptyset$. All maps in a limiting cone over a cofiltered diagram of epis are epi, if we assume the axiom of choice.

However, Choice and excluded middle do not help in making the mediator epi too. Consider the following chain diagram, in which each column denotes a set and the successive maps between the finite sets squash the top two elements:



Its limit is $\mathbb{N}^\top$, but there is also a cone of epis with vertex $\mathbb{N}$, but for which $\top$ is not in the image of the mediator, *i.e.* this is not surjective onto $\mathbb{N}^\top$.

Example 10.4 considers the analogous problem for pullbacks instead of limits of sequences. □

Remark 4.6 Venanzio Capretta, Tarmo Uustalu and Varmo Vene considered the categorical dual of our notion of well founded coalgebra, which they called an ***antifounded algebra*** [CUV09]. They presented a number of illuminating counterexamples that falsify our main recursion theorem unless we put other conditions on the category. Their simplest example is that $\mathsf{suc} : T\mathbb{N} \to \mathbb{N}$ is an antifounded algebra for $T \equiv \mathsf{id} : \mathbf{Set} \to \mathbf{Set}$, but there is no homomorphism from the trivial coalgebra $\mathsf{id} : \mathbf{1} \to T\mathbf{1}$, because its value should be the fixed point of suc, which we would like to be $\top$ in $\mathbb{N}^\top$. It would be instructive to compare their other counterexamples to our proofs, to see the necessity of the conditions in this section. □

These examples show that the filtered union condition is necessary, but infinitary colimits will only play a background role in this paper. This is why we do not give a formal account of the next issue:

Definition 4.7 A category is ***well powered*** if, for each object X, there is a "set" of isomorphism classes of monos $U \hookrightarrow X$.

On the face of it, the word "set" is an embarrassment, given that we aim to eliminate Set Theory from mathematical foundations. But, as mathematicians, we pay our words extra to mean what *we* want them to mean [Car72, Chapter 6]. In general, we do this by specifying the ways in which we intend to *use* the words, *i.e.* the axioms.

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A "set" of objects is not a chaotic jumble but a *single* object that is *dependent* on some *parameter*. In the geometric tradition, this arose as the object (such as a tangent space) *varied* from one place to another in a space. In type theory (and indeed longstanding symbolic usage in real analysis), it simply means a formula containing an unknown.

What we require of dependency is just to be able to *substitute* other formulae for the unknown parameter. This parameter has a certain *type*. Such types and their formulae form a category $\mathcal{S}$, called the ***base***, which may be **Set**, an elementary topos or even something simpler. Then, for each type Γ in $\mathcal{S}$, the objects whose parameter is of type Γ together form the ***fibre*** over Γ.

Substitution of a formula for a parameter (or along a morphism f) is an operation f^* on dependent objects. There are two techniques for capturing how f^* takes one fibre to another:

(a) if we consider the fibres as *separate* structures, they are the object part and f^* is the morphism part of a *functor* that is contravariant in f, giving an ***indexed*** structure; but

(b) the fibres may be combined into a single structure, called a ***fibration***, in which f^* acts by *pullback*.

The account that develops well-poweredness in most detail, in the indexed style, is [PS78] , although its goal is the adjoint functor theorem rather than our needs. The indexed approach has to contend with choices of isomorphic objects, which the fibred one avoids, but at greater learning cost. Brief accounts in the fibred style are in [Joh02, Example B1.3.14] and [Str05, §11]. Unfortunately, both techniques have rather obscure notation and huge diagrams, so, since we already have some very complicated ones, we will content ourselves a verbal description of how they work.

Definition 4.8 A ***generic*** object G is a parametric one that has the universal property that any *particular* object P is obtained as $P \cong f^*G$, by substitution of a value for the parameter in the generic one, The morphism f that achieves this is called the ***name*** of P.

The *type* of names is an object of the base category $\mathcal{S}$ and the generic object belongs to the fibre over this type. In particular, when the "objects" in question are monos $i : U \rightarrowtail X$ targeted at a particular object X of $\mathcal{C}$, the type of names is called $\mathsf{Sub}\,(X)$ and the generic subobject of X belongs to the fibre over this type.

Using the definition of genericity, any external structure that respects substitution induces an ***internal structure*** on the type of names in $\mathcal{S}$. For example, triangles of monos in $\mathcal{C}$ give rise to an internal order on $\mathsf{Sub}\,(X)$. In this sense we say that the external structure is ***equivalent*** to an internal one.

It is instructive to draw a few of these diagrams to show how, for example, pullbacks in $\mathcal{C}$ yield meets in $\mathsf{Sub}\,(X)$, making it an internal semilattice in $\mathcal{S}$. Then you will see that $\mathsf{Sub}\,(X)$ is like the handle of a marionette, with manoeuvres linked to the actions of the doll. With practice, we can just describe what the doll does, so long as we remember *how* it does it. We don't write out the diagram of strings because it conveys comparatively little information per cm^2 and is not really needed. In fact, the doll is *well powered* exactly when it is *impotent*, being able to do no more nor less than what the puppeteer makes it do.

Nevertheless, there is perhaps a PhD in collecting the applications of well powered categories from the literature and formalising results such as Proposition 4.10. This account would be analogous to those by Osius and others on the logic of a topos; indeed the subobject classifier provides the generic mono in a topos. We do not use universes or types of types here (Remark 1.13), but they can be presented categorically in a similar way, although without uniqueness of names.

Corollary 4.9 Any construction on a generic object that respects substitution corresponds uniquely to a morphism of the base category. In particular, the construction of one subobject of X from another corresponds to an endomorphism of $\mathsf{Sub}\,(X)$.

Proof An operation on a parametric object yields another object with the same parameter, *i.e.* in the same fibre, whilst binary operations such as categorical products combine the parameters using pullbacks in $\mathcal{S}$. We then use the universal property of the *generic* object of the resulting kind to define the morphism of the base category. $\square$

So far we have only discussed *finitary* structure such as composition and pullback. The original reason for requiring a "small" set of subobjects was so that we could legitimately form their union.

Proposition 4.10 External $\mathcal{S}$-indexed unions in a well powered category $\mathcal{C}$ correspond to joins in $\mathsf{Sub}\,(X)$.

Proof Any of the accounts of indexed and fibred categories explains how they handle colimits. Of course the "set" of objects of which we form the colimit is a single parametric one as before. In fact, the union operation is left adjoint to substitution and has an even simpler characterisation in that the opposite of the fibration functor is also a fibration.

The universal property of the generic subobject translates this into a join in the internal poset $\mathsf{Sub}\,(X)$. $\square$

Remark 4.11 Pataraia's Theorem 2.4 is for *internal* ipos in $\mathcal{S}$. The role of the union and well powered conditions that we have described is to provide an *equivalence* amongst external colimits, external unions of subobjects and internal joins. The same link also relates constructions in $\mathcal{C}$ to morphisms between objects of $\mathcal{S}$. In particular, the "relative successor" that we construct in the category in Constructions 5.2ff and 6.5 corresponds to a monotone inflationary endofunction of the internal ipo.

This has a fixed point by Pataraia's Theorem, which is valid precisely because the well powered condition turns colimits into joins in the object $\mathsf{Sub}\,(X)$ that is an internal poset in a topos. We translate this back into the category, as an object on which the construction yields an isomorphic object. $\square$

Remark 4.12 There is yet another reason why we need a "set" of subobjects, namely to justify universal quantification over them as predicates. (In Set Theory this distinction is known as ***unbounded*** *versus* ***bounded quantification***.)

When we introduced well-foundedness in Definitions 1.1 and 3.5, we called it a *scheme*, which means a property that we assert for *each individual* predicate ϕ. We will develop the *general* theory of well-foundedness in this way.

On the other hand, when we come to *apply* well-foundedness in the proof of our main theorem, we need it to be a *single* legitimate property in the logic of an elementary topos. For this it cannot be a scheme but must be *quantified* over all predicates ϕ.

Once again, by a "set" of predicates we mean a single generic predicate with a parameter. Well-foundedness with respect to a particular predicate ϕ is expressed in $\mathsf{Sub}\,(X)$ as above, with a parameter ϕ. Universal quantification over ϕ is now the *right* adjoint to substitution for ϕ, as is amply explained in the topos literature, *cf.* Notation 3.2. $\square$

Notation 4.13 We now turn to investigating the classes of "inclusions" that we might use in place of plain categorical monos when applying our ideas to objects with richer structure than sets have. We will use inclusions for three purposes in this paper:

(a) as the extents of *predicates* that test well-foundedness;

(b) as the inclusions of subcoalgebras that are the *supports* of attempts; and

(c) as the structure maps of *extensional* coalgebras.

All supports must be predicates to prove totality of recursion (Lemma 5.5), whilst supports and extensionality are thoroughly mixed up in Construction 7.11, so we must treat these as the same thing. Besides this, the predicates testing well-foundedness must have the cancellation property for monos because we want the existence of a splitting to be enough to deduce that they are isomorphisms.

Therefore we potentially have *two* classes of inclusions, one contained in the other, and both being contained in the split monos. We write

$$\rightarrowtail \text{ for predicates} \quad \text{and} \quad \hookrightarrow \text{ for supports and extensionality.}$$

It turns out that supports are always coalgebra homomorphisms, so we actually write $\hookrightarrow$ for them, whilst predicates just belong to the base category $\mathcal{C}$ and need not necessarily be coalgebra homomorphisms.

It is tempting (thinking in terms of so-called Descriptive Set Theory) to call $U \rightarrowtail X$ a *subspace* and $U \hookrightarrow X$ an *open* subspace of X. Unfortunately, this need not be the same as an open subspace in whatever topology the object X might carry.

Beware that these two classes of monos are *additional structure* for the situation, along with the category $\mathcal{C}$ and functor T. Varying the class of initial segments (and its orthogonal class of cofinal maps) will be important in future work [Tay25c, Tay25a]. On the other hand, we seem to be at liberty to choose the predicates in whatever way yields the optimum results, although we will then need to show that these are independent of the choices.

Remark 4.14 As you will see in the next section, we have some conflict in the objectives for this paper between proving the central recursion theorem and developing the whole theory of well founded coalgebras. For the general theory, we might typically want
(a) a large class of predicates so that we can make liberal use of induction, but
(b) a small class of supports.

For the proof of the recursion theorem, it turns out that
we only need to do induction over the supports,
so the two classes are the same.

We simply need a usable class of extensional well founded coalgebras that contains the iterates of the functor T applied to the initial object, as tightly as possible.

Therefore, in the particular application category, we would like to find some notion of inclusion that is both tractable and restrictive.

It is straightforward to substitute these chosen inclusions for the "monos" in the definitions above of unions and being well powered. However, Proposition 4.4 only works for plain monos and so needs to be replaced with some other argument, which is why we formulated Definition 4.3 instead of just asking that colimits be stable under pullback.

It may be possible to control the unions even further, such as by making the diagrams computable in some sense, using techniques from various forms of synthetic domain theory, but we leave that for another day.

For the general theory we do distinguish between the classes and so need to axiomatise them separately. In doing this, it is convenient to make an auxiliary definition for the closure conditions that are common to both classes:

Definition 4.15 A ***class of*** T***-monos*** $\mathcal{M}$ must
(a) contain all isomorphisms;
(b) contain all maps from the initial object (*cf.* Remark 4.2);
(c) be closed under composition;
(d) be preserved by the functor T ;
(e) be preserved by pullback along T-coalgebra homomorphisms; and
(f) satisfy the cancellation property for plain monos, $\forall f g.\ \ f\,;i = g\,;i \Longrightarrow f = g$.

Another easy but useful property that is also known as ***cancellation*** may be deduced as a "warm-up" exercise in the kind of diagram-chasing that we shall use throughout this paper:

Lemma 4.16 For any class of T-monos $\mathcal{M}$,
(a) if $i\,;m \in \mathcal{M}$ and m is a plain mono then $i \in \mathcal{M}$ too; and
(b) in the broken pullback for the induction premise (Definition 3.5), if the predicate $U \stackrel{i}{\rightarrowtail} X$ belongs to $\mathcal{M}$ then so does $H \stackrel{j}{\rightarrowtail} U$.

Proof The maps id, $(i\,;m)$, i and m form a pullback square. □

We are now ready to state the conditions for the two classes separately:

Assumption 4.17 The maps $\rightarrowtail$ used for predicates form a class of T-monos $\mathcal{M}$ that includes all inclusions of initial segments $\hookrightarrow$. However, whilst initial segments must be coalgebra homomorphisms, general predicates need not be.

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For additional results beyond the main recursion theorem,
(c) the class could include all regular monos (equalisers, *cf.* Lemma 6.4);
(d) the functor T could preserve inverse image diagrams (Section 9); or
(e) the inverse image operators f^* applied to predicates could have right adjoints f_*.

Recall that, in categorical logic, inverse images correspond to substitution, equalisers to equations, composition of monos to existential quantification and the right adjoint f_* to universal quantification, *cf.* Notation 3.2.

The conditions above are therefore natural and very flexible for considering precise restrictions on the logical strength of the predicates over which we may perform induction. This is possible (contrary to what was said in [Tay96b, Prop 6.7]) because we are making a distinction between the roles of predicates and initial segments. However, this direction is not explored in this paper; when it is, the claims that we make that results are independent of the choice of class of predicates should be re-examined carefully.

Remark 4.18 Suppose that the class $\mathcal{M}$ has a ***dominance*** $\top : \mathbf{1} \rightarrowtail \Sigma$ [Ros86]. This means a map of which every $\mathcal{M}$-map is the inverse image along a unique map, like Ω for all monos in a topos and the Sierpiński space Σ for open inclusions of topological spaces. Then Proposition 3.10 specialises to $\mathcal{M}$ with $\Theta \equiv \Sigma$. □

Now we turn to the other use of "monos" in the theory.

Assumption 4.19 The maps $\hookrightarrow$ used for inclusions of subcoalgebras and for structure maps of extensional coalgebras must form a class of T-monos (Definition 4.15) that
(a) is contained in the class of used for predicates;
(b) admits directed unions (Definitions 2.1 and 4.3); and
(c) is well powered (Definition 4.7).

Again, for additional results we may also assume that
(d) this class is part of a factorisation system (Section 8); or
(e) these monos admit pushouts that are unions (Section 10).

Remark 4.20 Since the monos in this class arise as composites of inclusions and structure maps of coalgebras, they are always homomorphisms, whereas predicates define monos in the underlying category. We will call them ***initial segments***, to exploit the intuition from ordinals. We write them as $\hookrightarrow\!\!\!\triangleright$, where two ends of the arrow signify two aspects:
(a) the triangle arrow*head* ($\longrightarrow\!\!\!\triangleright$) says that the map is a coalgebra homomorphism, which captures the *traditional* order-theoretic ideas (*cf.* Lemma 3.3); whilst
(b) the hook *tail* ($\hookrightarrow$) says that the underlying $\mathcal{C}$-map belongs to a special class of monos: this aspect is a *novelty* in this paper.

Examples 4.21 We will exploit the flexibility of using different classes of monos in our investigation of constructive ordinals [Tay25c]. There we will consider three classes in the category **Pos** of posets and monotone functions, on which the "lower-sets" functor $\mathcal{D}$ plays the role of the powerset:
(a) plain monos, which are inclusions of arbitrary subsets with a possibly sparser order relation; we call this class $\mathcal{I}$, but $\mathcal{D}$ does not preserve it;
(b) regular monos, which are inclusions that carry the restriction of the order relation; this class is called $\mathcal{R}$, the functor $\mathcal{D}$ preserves $\mathcal{R}$-maps but not their inverse images and the pushouts are poorly behaved; and
(c) inclusions of lower subsets, again with the restricted order; now $\mathcal{D}$ preserves this class $\mathcal{L}$ and its inverse images and the pushouts are well behaved, but the orthogonal class is not well co-powered (*cf.* Warning 8.16).

5 Generating well founded coalgebras

After a lengthy introduction, we now begin the study of how the category of well founded coalgebras is built up. If you skipped the previous section, you may just take the underlying category $\mathcal{C}$ to be **Set** and both kinds of "monos" ($\rightarrowtail$ and $\hookrightarrow\!\!\!\triangleright$) to be 1–1 functions. However, this theory also applies, for example, when the monos belong to either of the classes $\mathcal{R}$ or $\mathcal{L}$ but not $\mathcal{I}$ in **Pos** (Example 4.21).

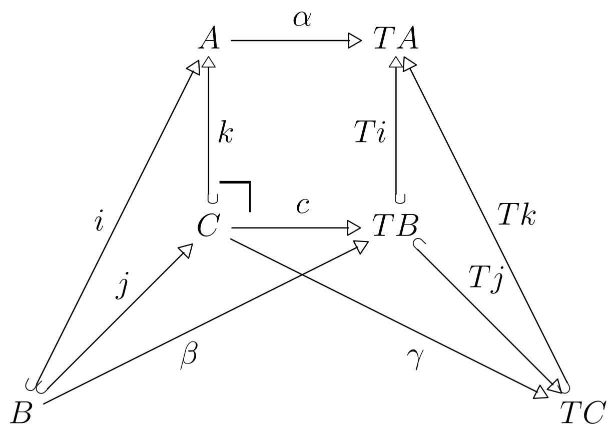
In the order-theoretic fixed point theorem, the directed joins play a background role. In any particular situation, most of the work goes into
(a) constructing the successor operation and
(b) proving maximality of fixed points for it.

So the first half of this section is about the successor operation on subcoalgebras that we introduced in Proposition 3.7. This has been called the ***next time operator*** elsewhere.

Lemma 5.1 The functor T preserves well founded coalgebras.

Proof This is special case of Lemma 5.8 with $c \equiv \mathsf{id}$, but it would be an instructive exercise to prove it directly. □

Construction 5.2 Let $i : (B, \beta) \hookrightarrow (A, \alpha)$ be a subcoalgebra. Then its ***relative successor*** $k : (C, \gamma) \hookrightarrow (A, \alpha)$ is given by pullback of α and Ti in $\mathcal{C}$.



The pullback mediator $j : B \to C$ makes $(B, \beta) \hookrightarrow (C, \gamma) \hookrightarrow (A, \alpha)$ as subcoalgebras (initial segments) when we define $\gamma \equiv c;Tj$. We will write sB for the relative successor (C); the operation s is inflationary because $j : B \hookrightarrow sB$.

Proof Since the functor T and inverse images preserve initial segments (Definition 4.15) and the latter obey the cancellation property (Lemma 4.16), if i is an initial segment then so successively are Ti, k, j, Tk and Tj. Finally, c, j and k are coalgebra homomorphisms, because

$$\gamma\,;Tc \;=\; c\,;Tj\,;Tc \;=\; c\,;T\beta$$
$$j\,;\gamma \;=\; j\,;c\,;Tj \;=\; \beta\,;Tj$$
$$k\,;\alpha \;=\; c\,;Ti \;=\; c\,;Tj\,;Tk \;=\; \gamma\,;Tk$$

□

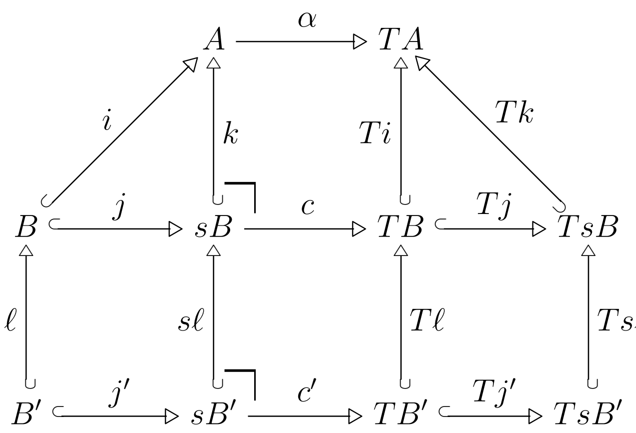
Lemma 5.3 (C, γ) is the pullback in **CoAlg**.

Proof In general, pullbacks of coalgebras are constructed by Proposition 9.5, which requires T to preserve the pullback in order to define the structure map γ, but in this case we already have it. Any cone $A \leftarrow D \to TB$ has a unique pullback mediator $h : D \to C$ in $\mathcal{C}$, so these two maps are $h\,;k$ and $h\,;c$. They are homomorphisms so long as

$$\delta\,;Th\,;Tk \;=\; h\,;k\,;\alpha \;=\; h\,;\gamma\,;Tk \quad \text{and} \quad \delta\,;Th\,;Tc \;=\; h\,;c\,;T\beta \;=\; h\,;\gamma\,;Tc,$$

but Tk is (plain) mono, so $\delta\,;Th = h\,;\gamma$, making h a homomorphism as required. □

Lemma 5.4 The relative successor construction s is monotone (functorial) in B.



Proof The diagram repeats the Construction for initial segments B' and B, where the square from $C \equiv sB$ to TA and rectangle from sB' to TA are both pullbacks. Applying functoriality and well known properties of pullbacks to the inclusion $B' \hookrightarrow B$, the rectangle is the composite of two pullback squares. $\square$

The next result provides maximality of fixed points for Pataraia's Theorem 2.4 and is actually the sole place in our proof of the recursion theorem where we use well-foundedness. This step could alternatively be seen as using the property of B that it is a well founded *element* of the ipo of sub*objects* of A (Proposition 3.7), with $w \equiv B$ and $x \equiv B'$ in Lemma 2.3.

Lemma 5.5 In the previous diagram, if (B, β) is well founded and (B', β') is fixed by the relative successor $(j' : B' \cong sB')$ then $\ell : B' \cong B$, *cf.* Lemma 2.3.

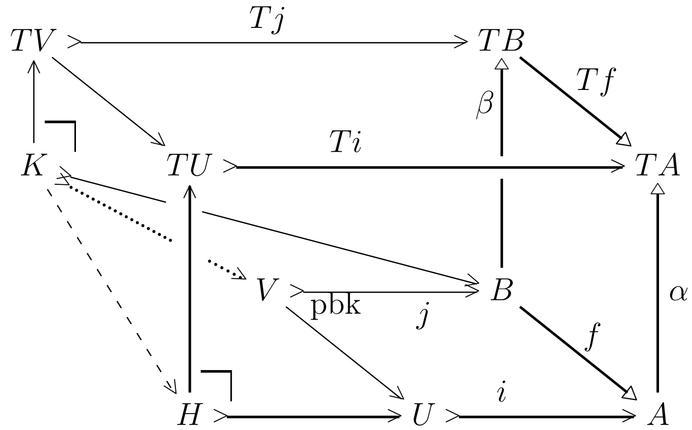
Proof The square from sB' to TB is a pullback, but when j' is an isomorphism and j is mono, the one from B' to TB' is also a pullback. It is the *tight* form ($K = U = B'$) of the one in Definition 3.5 of well-foundedness. Therefore $B' \cong B$.

Observe, however, that the monos in this diagram are marked as initial segments rather than predicates. This is why we required all initial segments to be predicates in Assumption 4.17. $\square$

This has already invoked the notion of well-foundedness, so we need to prove that it is preserved by the successor. In the case of the covariant powerset, any subcoalgebra of a well founded coalgebra is again well founded, by Proposition 1.4. Using this, we could deduce well-foundedness of $sB \equiv C$ from that of TB and hence from that of B by Lemma 5.1. However, since we have weakened the requirements on the functor, we need a more complicated proof.

First we give the categorical version of Lemma 2.14 about bisimulations, since Lemma 3.3 generalised them to coalgebra homomorphisms, whilst pullback captures substitution.

Lemma 5.6 The induction premise (broken pullback in Definition 3.5) is stable under pullback against coalgebra homomorphisms.



Beware that we are not yet saying anything about i being an isomorphism.

Proof The thick lines show the homomorphism $f : B \longrightarrow\!\triangleright\, A$ and the given induction premise $H \rightarrowtail U$ for the predicate $i : U \rightarrowtail A$.

Let $j : V \rightarrowtail B$ be the inverse image of i along f. Apply T to this pullback, to give the *parallelogram* at the top.

Form the inverse image $K \rightarrowtail B$ of Tj along β, giving the upper trapezium, with unbroken lines. This makes K the induction hypothesis for $V \rightarrowtail B$.

The top, back and right quadrilaterals commute (from K *via* A or TU to TA), so there is a pullback mediator $K \to H$ that makes the left and bottom quadrilaterals commute, *i.e.* from K to TU and to A. The map $K \to H$ deduces the induction hypothesis for U from that for V.

Because of this, there is a pullback mediator $K \to V$ that makes everything commute, in particular from K to B. Then $K \to V$ is the required induction premise. □

Remark 5.7 If the functor T preserves inverse images then the top parallelogram is a pullback: beware that we did *not* assume this. In this case, the lower left quadrilateral (K, V, U and H) is also a pullback. From this it follows that tightness is preserved: if $H \cong U$ then $K \cong V$. □

We use this technical result to prove the ***sandwich lemma*** that shows that the relative successor preserves well-foundedness.

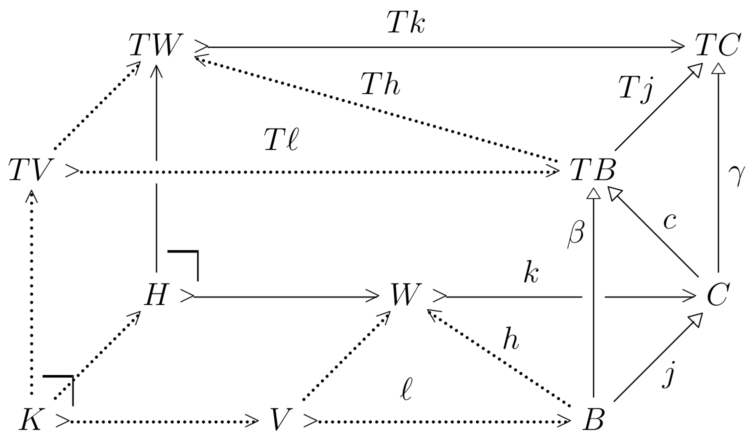
Lemma 5.8 Let (B, β) be a well founded coalgebra and $j : B \to C$ and $c : C \to TB$ maps such that $\beta = j\,;c$ (but not necessarily mono). Put $\gamma \equiv c\,;Tj$. Then (C, γ) is also a well founded coalgebra and j and c are homomorphisms.

Proof They are homomorphisms because, as in Construction 5.2,

$$j\,;\gamma \;\equiv\; j\,;c\,;Tj \;=\; \beta\,;Tj \qquad \text{and} \qquad \gamma\,;Tc \;\equiv\; c\,;Tj\,;Tc \;=\; c\,;T\beta.$$

Now let $k : W \rightarrowtail C$ satisfy the induction premise given by the broken pullback H at the back. Form the inverse image of this along j, *in the sense of* Lemma 5.6. Beware that this means that K is the pullback in the front rectangle but not necessarily in the lower left parallelogram.

This gives the induction premise K for the predicate $\ell : V \rightarrowtail B$:



Since B is well founded, $\ell : V \cong B$ and so there is a map $h : B \to W$ making the triangle with C commute. The one with TB, TW and TC also commutes.

The triangle on the right commutes too ($\gamma = c\,; Tj$), so the maps $C \to TB \to TW$ and $\mathsf{id} : C \to C$ form a commutative square at TC. This factors through the pullback H, splitting the inclusion $H \rightarrowtail W \rightarrowtail C$ as required [Tay96b, Lemma 8.2]. □

The proof of the Recursion Theorem 1.5 forms the *union* of attempts in part (h), so we turn to considering colimits next, but see Definition 4.3 for the relationship between colimits and unions in general categories.

Note, however, that we are not asking for *new* colimits: we are merely *enhancing* the properties of those that *already* exist in the category $\mathcal{C}$, by showing that the categories of coalgebras and of well founded coalgebras inherit them.

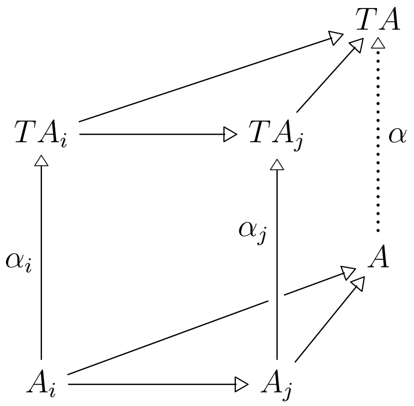
Although we state the Proposition for general colimits, we only use the initial object and directed unions (Definition 2.1) in our main proof of the recursion theorem. We will consider pushouts in Section 10.

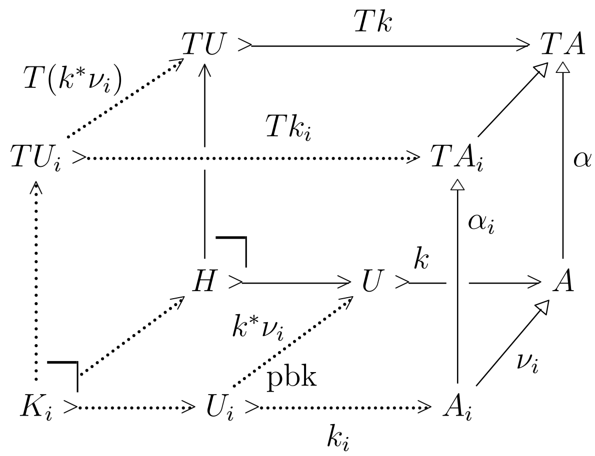
Lemma 5.9 The initial object $\emptyset$ of $\mathcal{C}$ carries a unique T-coalgebra structure, which is well founded and is the least subcoalgebra of any coalgebra.

Proof Easy, but *cf.* Theorem 1.5(a), Remark 4.2 and Definition 4.15(b,f). □

Proposition 5.10 The forgetful functors **WfCoAlg** $\longrightarrow$ **CoAlg** $\longrightarrow \mathcal{C}$ create colimits. That is, the colimit of any diagram of coalgebras and homomorphisms is given by the colimit of their carriers, if this exists, and then the structure map is uniquely determined. If the individual coalgebras are well founded then so is their colimit (*cf.* Theorem 1.5(b)).

Proof The structure map α on the colimit is the colimit mediator, as shown in the diagram on the left, where the colimiting cocone consists of coalgebra homomorphisms, *i.e.* the parallelograms from A_i to TA commute.





Now suppose that the α_i are well founded and let $k : U \rightarrowtail A$ be a predicate satisfying the induction premise for the colimit α (the back rectangle, from H to TA).

Use Lemma 5.6 to form the inverse images K_i of this induction premise against the homomorphisms $A_i \longrightarrow\!\!\triangleright\, A$ of the colimiting cocone. That is, the front rectangle is a pullback but the lower left parallelogram need not be.

Since each A_i is well founded, $k_i : U_i \cong A_i$.

Now U is the vertex of a cocone $\left(k_i^{-1}\,; k^*\nu_i : A_i \to U\right)$ over the diagram (A_i), so it has a mediator from the colimit A, and $i : U \cong A$ as required [Tay96b, Prop 6.6]. $\square$

Now we make a change of gear in the development. So far we have discussed successors and colimits of well founded coalgebras as *external* operations on the underlying category $\mathcal{C}$. The fixed point theorem in Section 2, on the other hand, works in **Set** (or a topos $\mathcal{S}$), in which the subcoalgebras form an *abstract* ipo. In particular, there needs to be a "set" (object of the topos) that encodes them *all*, which what *well-poweredness* means (Definition 4.8). Even if $\mathcal{C}$ is **Set**, it plays different roles, as the category in which we build coalgebras and as the one where the ipo lives.

This means that the foregoing discussion of *concrete* objects and morphisms in $\mathcal{C}$ should be repeated as one about their *names* as elements of an ipo in the topos. This can in principle be done using a fibration, where we called the relationship between the two a *marionette.*

Here are the things that well-poweredness provides:

Lemma 5.11 The categories of $\mathcal{C}$-subobjects, subcoalgebras and well founded subcoalgebras of any coalgebra (A, α), together with inclusions (initial segments) between them, are equivalent to $\mathcal{S}$-internal posets

$$\mathsf{WfSeg}\,(A,\alpha) \;\subset\; \mathsf{Seg}\,(A,\alpha) \;\subset\; \mathsf{Sub}\,(A).$$

Under this equivalence, the relative successor construction corresponds to an inflationary monotone endofunction s of each of the three posets.

The three posets are ipos, whose common least element and directed joins correspond to the initial object and directed unions.

There is a least fixed point of s, which corresponds to a well founded coalgebra. If A is well founded then this is it.

Proof Assumption 4.19, Lemma 5.9 and Proposition 5.10 provide the colimits in $\mathcal{C}$, **CoAlg** and **WfCoAlg**, but we needed Definition 4.3 to make these colimits agree with unions of subcoalgebras.

The well powered condition (Proposition 4.10) links the external unions with the internal joins.

It is used again to justify quantification over the class of predicates in the definition of well-foundedness (Remark 4.12).

Corollary 4.9 internalises the relative successor and also makes external isomorphisms correspond to internal equalities. $\square$

We can now formulate the idiom of induction that we require for our uses of Pataraia induction in the next two sections. It is similar to the way sets are built up in Zermelo's Set Theory [Zer08b], where the Sandwich Lemma corresponds roughly to subsets of powersets.

Theorem 5.12 Any property of coalgebras that holds for $\emptyset$ and is preserved by directed unions and sandwiching *à la* Lemma 5.8 is valid for all well founded coalgebras.

Proof Although this appears to be about the *class* of well founded coalgebras, it is a *scheme* of results about the goal (A, α) as usual, because we just require the predicate $\Phi(B)$ to be defined for $B \in \mathsf{Seg}\,(TA)$.

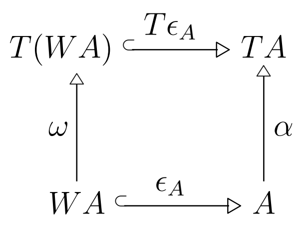
The sandwich property, that $\Phi(B)$ implies $\Phi(C)$ whenever C splits $\beta : B \to C \to TB$, means that the relative successor (Lemma 5.11) for initial segments of A preserves Φ. $\square$

This theorem is the analogue for well founded *coalgebras* of Corollary 2.7 for relations. The recursion theorem and subsequent developments will build on this and not directly on the definition of a well founded coalgebra. We will see in the next section how those are approximations to the initial algebra for a *functor*, so when this theory is generalised to algebraically more complicated situations such as type theories, it is this Theorem that will need to be reproduced.

The remainder of this section considers some variations on our definition of well founded coalgebras.

Remark 5.13 If T preserves inverse images then various quadrilaterals become pullbacks (Remark 5.7) and the results of this section also hold for tight well-foundedness. Using the induction principle in the tight case, tight and loose well-foundedness are then equivalent. It is far from clear whether this is the case, or even how to prove the tight analogue of Theorem 5.12, without the stronger assumption. $\square$

Proposition 5.14 Any coalgebra $A \xrightarrow{\alpha} TA$ has a greatest well founded subcoalgebra.



Proof Corollary 5.11 also defined the subipo $\mathsf{WfSeg}\,(A, \alpha) \subset \mathsf{Seg}\,(A, \alpha)$ of well founded subcoalgebras. By Lemma 5.8, the relative successor restricts to an endofunction of the smaller ipo, where it is inflationary and monotone. By Lemma 2.3 and Theorem 2.4 the sub-ipo of well founded elements has a greatest element and it is the unique fixed point. $\square$

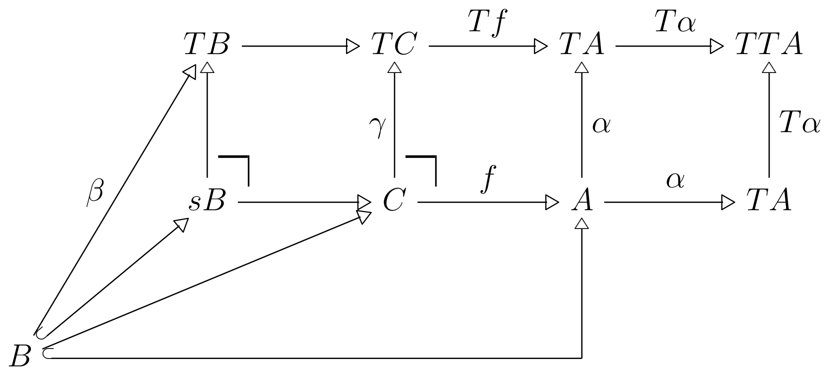
Proposition 5.15 This is independent of the choice of classes of predicates and initial segments.

Proof As an element of the larger ipo $\mathsf{Seg}\,(A, \alpha)$, WA is characterised as the *least* fixed point of the successor. The statement of this property is independent of the notion of well-foundedness: If we re-define the latter for a larger class of predicates that satisfies Definition 4.15, even though there may be fewer well founded subcoalgebras, WA is still one of them and the proof that it is the largest one also remains valid. □

Theorem 8.8 improves this *greatest* well founded subcoalgebra to a *right adjoint* or coreflection and Corollary 9.7 expresses it more simply as a pullback, both under additional assumptions.

The last variation gives a foretaste of the recursion theorem. It *apparently* eliminates "predicates" from well-foundedness altogether, but there must at least *exist* some system of them to prove the result. Beware that split monomorphisms only obey some of the parts of Definition 4.15, since non-trivial monos $\emptyset \to X$ in **Set** do not split.

Proposition 5.16 If split monomorphisms of coalgebras are predicates then any homomorphism $f : C \longrightarrow A$ (not necessarily mono) to a well founded coalgebra for which the (middle) square is a pullback is an isomorphism.



Proof We claim that any subcoalgebra $B \hookrightarrow A$ factors through C: This is trivial for $B \equiv \emptyset$. If it holds for B then the pullback defining sB (Construction 5.2) factors through the given one for C. The only fixed point is A and so by Theorem 5.12 there is a homomorphism $A \longrightarrow C$ such that $A \longrightarrow C \longrightarrow A$ is id_A. Composing this with α makes C a sandwich of A, so it is well founded by Lemma 5.8. If the split monomorphism $A \longrightarrow C$ is a predicate then it is an isomorphism. □

6 The recursion theorem

Now we are ready to prove the recursion theorem for well founded coalgebras. Since we only assume that the functor T preserves monos and not their inverse images, the proof is more subtle than the one in [Tay99, §6.3]. It is based on our *revised* argument for well founded relations in Theorem 2.8, which uses *directed* unions of attempts, instead of forming the union of *all* of them. In particular, we do not assume that we have *binary* unions, although we will return to them in Section 10.

The proof has similar components to those in the previous section, dealing with the empty attempt, successors, directed unions and maximality of fixed points for Pataraia's

Theorem. The particular novelty of our proof is the more careful analysis of the successor. Like the previous section, this one applies to the classes $\mathcal{R}, \mathcal{L} \subset \mathbf{Pos}$ but not $\mathcal{I}$ in Example 4.21.

Remark 6.1 An *attempt* from a coalgebra $\alpha : A \longrightarrow\!\!\!\triangleright\, TA$ to an algebra $\theta : T\Theta \to \Theta$ is intended to be a partial map $f : A \rightharpoonup \Theta$ that is a subhomomorphism in the sense that

$$\begin{array}{ccc} TA & \xrightarrow{\ Tf\ } & T\Theta \\ \uparrow{\scriptstyle\alpha} & \sqsubseteq & \downarrow{\scriptstyle\theta} \\ A & \xrightarrow{\ f\ } & \Theta \end{array}$$

i.e. if the composite *via* TA is defined then so is the direct map to Θ and then they are equal, *cf.* the definition in Theorem 1.5.

Composition of partial functions in a category uses inverse images. In order to define a category of coalgebras and *partial* homomorphisms, the functor T should therefore *preserve* inverse image diagrams, as the powerset and term algebra functors do.

However, the structure maps α and θ are total and we never need to compose partial maps. The notion of attempt therefore has a simple equivalent form that is sufficient to carry out the proof of the theorem:

Definition 6.2 An ***attempt*** from A to Θ is a diagram of the form

$$\begin{array}{ccccc} TA & \xleftarrow{\ Ti\ } & TB & \xrightarrow{\ Tf\ } & T\Theta \\ \uparrow{\scriptstyle\alpha} & & \uparrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\theta} \\ A & \xleftarrow{\ i\ } & B & \xrightarrow{\ f\ } & \Theta \end{array}$$

That is, a subcoalgebra inclusion (initial segment) $i : B \hookrightarrow A$ together with coalgebra-to-algebra homomorphism $f : B \longrightarrow\!\!\!\triangleright\, \Theta$. A map f satisfies the recursion scheme (Definition 3.8) exactly when it is a ***total attempt***, with $i : B \cong A$.

We also need a well powered assumption for attempts, which is easily adapted from that for initial segments (Definition 4.7ff). Alternatively we may consider them as subobjects of $A \times \Theta$ instead of those of A. Then, for any given coalgebra and algebra, there is a set or $\mathcal{S}$-object $\mathsf{Att}\,(A, \alpha, \Theta, \theta)$ of attempts from A to Θ, *cf.* $\mathsf{Seg}\,(A, \alpha)$ in Corollary 5.11.

Lemma 6.3 There is a "support" function (morphism of $\mathcal{S}$)

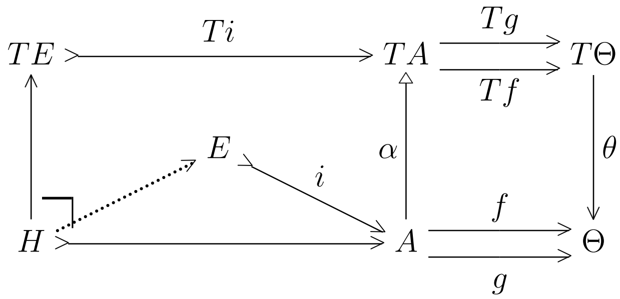
$$\mathsf{supp}\ : \mathsf{Att}\,(A, \alpha, \Theta, \theta) \longrightarrow \mathsf{Seg}\,(A, \alpha) \quad \text{by} \quad (A \xleftarrow{\ i\ } B \xrightarrow{\ f\ } \Theta) \longmapsto (B \xhookrightarrow{\ i\ } A).$$

Proof An application of Corollary 4.9. □

One way to show that attempts are *unique* is by an easy application of well-foundedness. The corresponding result for well founded relations typically appears like this in set theory textbooks, separately from the main *existence* part of the recursion theorem.

Lemma 6.4 Let A be a well founded coalgebra, Θ an algebra and $f, g : A \rightrightarrows \Theta$ be total attempts. Then $f = g$ (*cf.* Theorem 1.5(d)).

Proof The two parallel squares on the right commute since f and g are total attempts. Let $i : E \rightarrowtail A \rightrightarrows \Theta$ be the equaliser in $\mathcal{C}$.



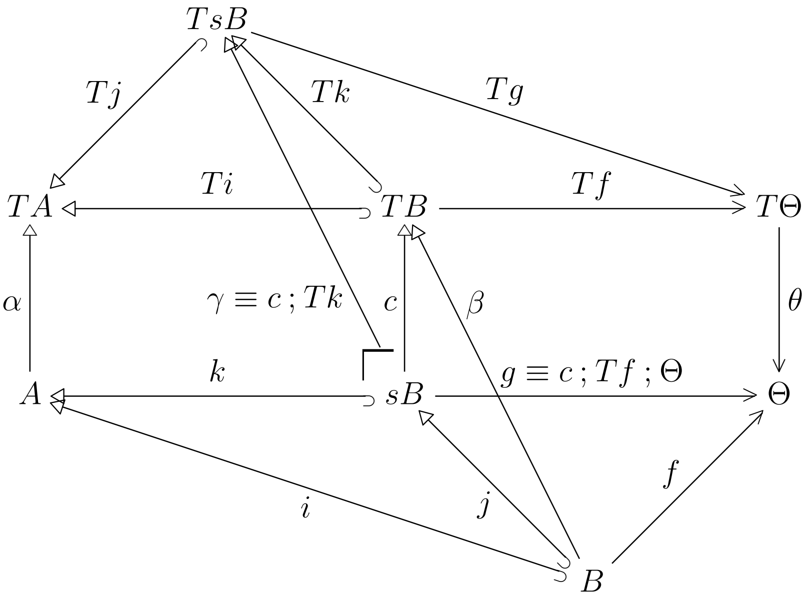
Form the pullback H of $A \to TA \leftarrow TE$; the composites $H \rightrightarrows T\Theta$ are equal by construction, so those $H \rightarrowtail A \rightrightarrows \Theta$ are also equal. Then $H \rightarrowtail A$ factors through the equaliser, so $H \rightarrowtail E \rightarrowtail A$. Hence $i : E \cong A$ by well-foundedness of A and so $f = g$. [Mik22, page 67] [Osi74, Prop 6.5] [Osi75, Prop 6.3] [Tay96a, 2.5] [Tay96b, Prop 6.5] [Tay99, Prop 6.3.9]. □

You will object that we did not ask for equalisers in Section 4, either in the category itself or in the class of predicates over which we may perform induction. This lemma is valid in **Set**, and also in **Pos** if we use $\mathcal{R}$ for predicates, but not using $\mathcal{L}$ (Example 4.21). However, von Neumann's *original* argument [vN23, vN28] included uniqueness in the induction hypothesis. Following his example, we obtain a more subtle version of the proof that uses the structure of subcoalgebras and does not need equalisers after all, so it is valid for $\mathcal{L}$:

Lemma 6.5 There is a bijection between attempts

$$A \xleftarrow{\;i\;} B \xrightarrow{\;f\;} \Theta \qquad \text{and} \qquad A \xleftarrow{\;j\;} sB \xrightarrow{\;g\;} \Theta,$$

where sB is the relative successor of B (Construction 5.2). Hence the successor lifts not only the existence but also the uniqueness of an attempt.



Proof Let $(A, \alpha) \overset{i}{\hookleftarrow} (B, \beta) \xrightarrow{f} (\Theta, \theta)$ be an attempt, so

$$i \,;\alpha \;=\; \beta\,; T\alpha \qquad \text{and} \qquad f \;=\; \beta\,; Tf\;\theta$$

Then the relative successor attempt is defined by

$$\gamma \;\equiv\; c\,; Tj \qquad \text{and} \qquad g \;\equiv\; c\,; Tf\,;\theta$$

and satisfies

$$\begin{aligned} f &= \beta : Tf\,;\theta \;=\; j\,;c\,;Tf\,;\theta \;=\; j\,;g \\ g &\equiv c\,;Tf\,;\theta \;=\; c\,;Tj\,;Tc\,;TTf\,;T\theta\,;\theta \\ &= c\,;Tj\,;Tg\,;\theta \;\equiv\; \gamma\,;Tg\,;\theta. \end{aligned}$$

So $(A, \alpha) \overset{j}{\hookleftarrow} (sB, \gamma) \xrightarrow{g} (\Theta, \theta)$ is also an attempt, extending f.

Conversely, $f \equiv i\,;g$ satisfies

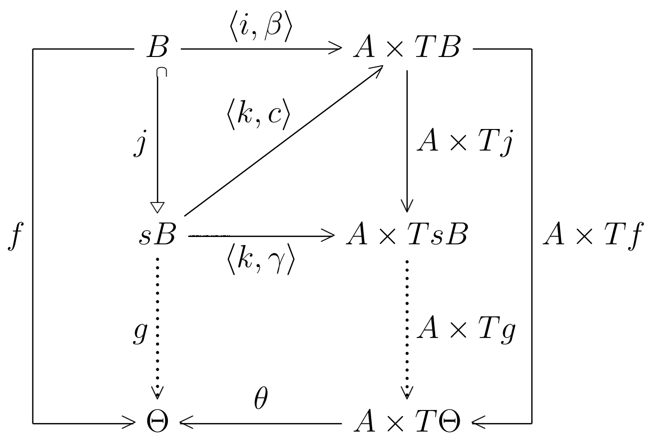
$$\begin{aligned} f &\equiv j\,;g \;=\; j'\gamma\,;Tg\,;\theta \;=\; j\,;c\,;Tj\,;Tg\,;\theta \;=\; \beta\,;Tf\,;\theta \\ g' &\equiv c\,;Tf\,;\theta \;=\; c\,;Tj\,;Tg\,;\theta \;=\; \gamma\,;Tg\,;\theta \;=\; g, \end{aligned}$$

establishing the bijection. □

The parametric version is similar and is the only place where we use binary products; to make these into coalgebras would require a "strength" for the functor, $TA \times TC \to T(A \times C)$.

Lemma 6.6 There is a bijection between parametric attempts given by

$$g \;\equiv\; \langle k, (c\,;Tf)\rangle\,;\theta \qquad \text{and} \qquad f \;=\; j\,;g.$$ □



Lemma 6.7 The initial object is the support of a unique attempt. For any directed diagram of subcoalgebras, if each member is the support of a unique attempt then so is the union of the diagram.

Proof The statements are the universal properties of the initial object and filtered colimits, but we need to say that they are unions (Definition 4.3). Also *cf.* Remark 4.2, Lemma 5.9 and Proposition 5.10. □

We can now achieve our principal goal, the ***Recursion Theorem***, based on Theorem 2.8 for well founded relations.

Theorem 6.8 From any well founded coalgebra (A, α) to any algebra (Θ, θ) there is a unique total attempt or coalgebra-to-algebra homomorphism.

Proof This is an application of Theorem 5.12, *i.e.* Pataraia induction over the ipo $\mathsf{Seg}\,(A, \alpha)$.

There the relative successor defined an endofunction of Seg whose unique fixed point is the top element, A itself. Similarly, Lemma 6.5 defined an endomorphism of $\mathsf{Att}\,(A, \alpha, \Theta, \theta)$ and a map $\mathsf{supp} : \mathsf{Att} \to \mathsf{Seg}$ that commutes with the two successors (Corollary 4.9). This situation is wholly about objects and morphisms of the topos $\mathcal{S}$.

Now consider the predicate $\Phi(B)$ on the initial segments $i : B \hookrightarrow A$ that says that there is a *unique* attempt with support B. That is,

$$\Phi(B) \quad \equiv \quad \exists! a \in \mathsf{Att}\,(A, \alpha, \Theta, \theta).\, \mathsf{supp}\,(a) = B.$$

Then $\Phi(\emptyset)$ and Φ is preserved by directed unions by Lemma 6.7 and by the successor by Lemma 6.5.

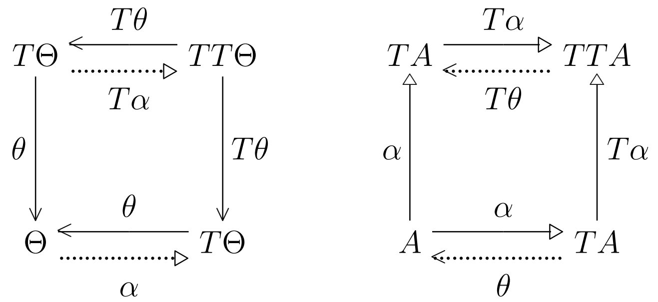
Hence, by the induction principle, $\Phi(A)$ holds. This says that there is a unique attempt with support A, *i.e.* a total one, or a solution to the recursion equation [Mik22, pp 68–70] [Osi75, Prop 6.5] [Tay99, Thm 6.3.13] □

The *statement* of the Theorem is independent of the notion of initial segment that we choose. Also, if we enlarge the class of predicates then there are just fewer well founded coalgebras and the result remains the same.

We have developed the theory of well founded *co*algebras to approximate the initial *algebra* when the functor T does not have one, such as in the case of the powerset. When the initial algebra does exist, we therefore need to link the two accounts together.

Two of the steps in the circular equivalence below are based on observations by Joachim Lambek [Lam68] and by Daniel Lehmann and Michael Smyth [LS81, §5.2]. Lambek discusses systems of coherently commuting functors and gives a criterion for the existence of a common fixed point.

Proposition 6.9 The structure maps of the initial algebra, final coalgebra and final well founded coalgebra, if they exist, are isomorphisms.

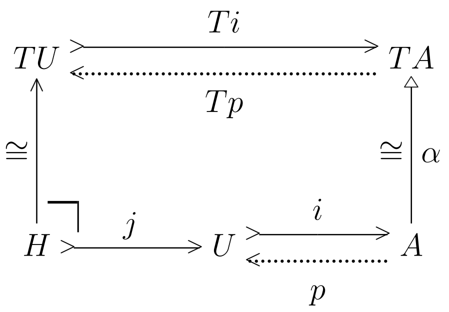


These objects are therefore both algebras and coalgebras and we call them ***fixed points*** of the functor. Coalgebra-to-algebra homomorphisms from or to them are respectively the same as plain algebra or coalgebra homomorphisms.

The successor relative (Lemma 5.2) to the initial algebra is just the functor T.

Proof This is illustrated by the diagrams. It also applies to the final *well founded* coalgebra because the functor T preserves well-foundedness by Lemma 5.1. In Lemma 5.2, since $A \cong TA$ also $C \cong TB$. □

Proposition 6.10 The initial algebra A is well founded *quâ* coalgebra.



Proof Since the structure map is invertible, so is its pullback, so $TU \cong H \overset{j}{\rightarrowtail} U$ makes U an algebra and $i : U \rightarrowtail A$ an algebra monomorphism. But this is split since A is initial *quâ* algebra. Hence A is well founded *quâ* coalgebra. ☐

Corollary 6.11 If any of the following exists then it satisfies the other properties too:
(a) a final well founded coalgebra;
(b) a well founded coalgebra whose structure map is an isomorphism;
(c) an initial fixed point;
(d) an initial algebra.
Moreover, it is unique up to unique isomorphism.

Proof [a⇒b and c⇔d] Proposition 6.9. [d⇒b] Proposition 6.10.

[b⇒a] Let (B, β) be a well founded coalgebra with β invertible, so (B, β^{-1}) is an algebra. Then for any well founded coalgebra (A, α) the recursion theorem gives a unique homomorphism $A \longrightarrow\!\!\triangleright\, B$. This is the coalgebra homomorphism that makes B terminal.

[b⇒c] The Recursion Theorem 6.8 says that the final well founded coalgebra has the universal property of the initial algebra. ☐

Corollary 6.12 Any algebra or coalgebra homomorphism to the initial algebra whose square is a pullback is an isomorphism.

Proof This is Proposition 5.16 for the terminal well founded coalgebra. Even though it *looks* as simple as Lambek's Lemma, this and the sufficiency of (b) in the previous result depend on the whole of the machinery that we have developed. ☐

Corollary 6.13 If T has a final coalgebra F then its greatest well founded subcoalgebra $A \equiv WF$ is the initial algebra.

Proof The structure map of F is an isomorphism, so by cancellation of monos, that of A is mono too. But TA is another well founded subcoalgebra of F, so $A \cong TA$, whence A is the initial algebra. ☐

This is implicit in Peter Aczel's work on non-well-founded sets [Acz88, Acz89]. That there is a homomorphism $A \to F$ follows from Lambek's lemma and his paper says more about the category of fixed points, in which A and F are initial and final. However, the fact that $A \to F$ is mono must depend on the assumptions in Section 4. Peter Freyd considered the situation where $A \cong F$, which he called ***algebraic compactness*** [Fre91].

Theorem 7.7 and Corollary 9.7 have more to say about the initial object.

7 Extensionality

Now that we have some understanding of the Axiom of Foundation generalised to coalgebras, we turn to the Axiom of Extensionality.

Definition 7.1 A coalgebra $\alpha : A \longrightarrow TA$ is ***extensional*** if α is an initial segment.

That is, it belongs to the same class of monos that we used for subcoalgebras in our proof of the recursion theorem (Assumption 4.19). If you skipped Section 4, you should simply understand α to be mono, *cf.* Definition 1.7 when $T \equiv \mathcal{P}$, but these results are also applicable for $\alpha \in \mathcal{R}$ or $\mathcal{L} \subset$ **Pos** (Examples 4.21).

An ***ensemble*** is an extensional well founded coalgebra. The name could be qualified by stating the category, functor and two classes of monos that are used in the definition, especially the one used for initial segments. The name is justified by the similarity that we will demonstrate in this section between their behaviour and that of transitive sets in Set Theory (Remark 1.10), being fragments of the initial T-algebra. The latter doesn't exist in the original situation $T \equiv \mathcal{P}$ but could for other functors.

We write **Ens** $\subset$ **ExtCoAlg** $\subset$ **CoAlg** for the full subcategories of ensembles and extensional coalgebras, with coalgebra homomorphisms.

Gerhard Osius used the name *transitive set object* (*cf.* Remark 1.10) for our $\mathcal{P}$-ensembles and defined a general *set-object* to be an $\mathcal{S}$-subobject of one of these, but not necessarily a subcoalgebra [Osi74, Def. 7.14]. From this he developed Set Theory along the lines of Zermelo, in fact giving a logical subtopos of $\mathcal{S}$. See [Tay96a, §3] for another account of this, but we won't go any further with it here.

Lemma 7.2 Any initial segment of an extensional coalgebra is extensional.

$$\begin{array}{ccc} TB & \xrightarrow{Ti} & TA \\ \uparrow{\scriptstyle\beta} & & \uparrow{\scriptstyle\alpha} \\ B & \xrightarrow{i} & TA \end{array}$$

Proof Ti and β are in $\mathcal{M}$ because it is closed under T, composition and cancellation (Lemma 4.16). □

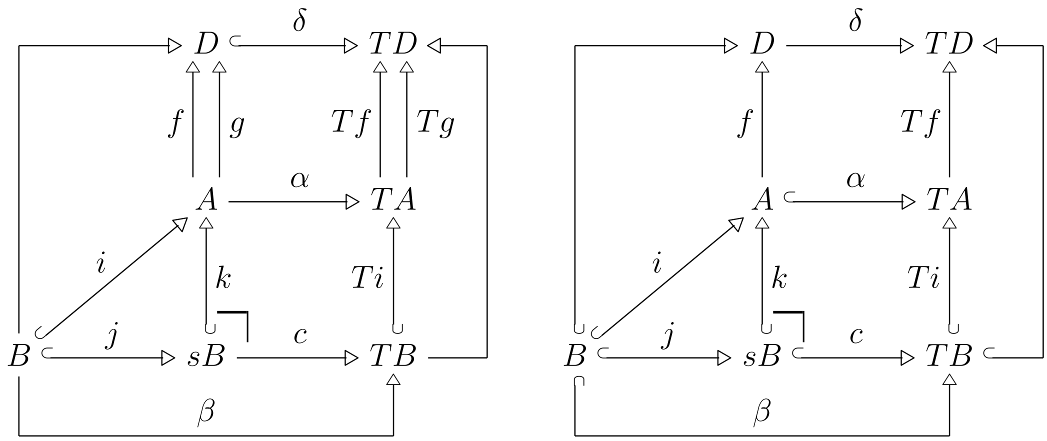
The next result shows the effect of extensionality on the source and target of coalgebra homomorphisms, separately. These are further applications of Pataraia induction over the initial segments B of the well founded coalgebra A.

Lemma 7.3 Let $\alpha : A \longrightarrow TA$ and $\delta : D \longrightarrow TD$ be coalgebras, with A well founded.

(a) If D is extensional then there is at most one homomorphism $A \longrightarrow D$; for this we only require $\delta : D \to TD$ to be plain mono, not all of Assumptions 4.19 or even that $T\delta$ also be mono.

(b) If A is $\mathcal{M}$-extensional then any homomorphism $f : A \longrightarrow D$ is in $\mathcal{M}$; this requires a stronger cancellation property than we have stated, namely $h\,;\delta \in \mathcal{M} \Rightarrow h \in \mathcal{M}$.

When both A and D are extensional, the fact that δ is plain mono in part (a) suffices to give the cancellation property for $\mathcal{M}$ in part (b), by Lemma 4.16. Hence the category **Ens** is a preorder and its morphisms belong to the class $\mathcal{M}$.



Proof As in Lemma 6.7, the properties hold for $B \equiv \emptyset$ and are preserved by directed unions because of Assumption 4.19, in particular that colimit mediators are in $\mathcal{M}$ in part (b).

The successor operation that gives sB was defined in Construction 5.2. Here are the induction steps for the two predicates $\Phi(B)$:

(a) Suppose that $i\,;f = i\,;g : B \longrightarrow D$. Then $Ti\,;Tf = Ti\,;Tg$ and

$$k\,;f\,;\delta \;=\; k\,;\alpha\,;Tf \;=\; c\,;Ti\,;Tf \;=\; c\,;Ti\,;Tg \;=\; k\,;\alpha\,;Tg \;=\; k\,;g\,;\delta,$$

so the diagram commutes from sB to TD. But δ is (plain) mono, so $k\,;f = k\,;g : sB \longrightarrow D$.

(b) Suppose that $B \hookrightarrow A \longrightarrow D$ is mono (in $\mathcal{M}$). Since the class is closed under T and composition and the whole diagram commutes, $sB \longrightarrow TD$ is also mono (in $\mathcal{M}$). If $\mathcal{M}$ has the stronger cancellation property then $sB \longrightarrow D$ is in $\mathcal{M}$ too.

Hence these properties hold for the fixed point $B \equiv A$. □

Examples 7.4

(a) Each of the parts holds for 1–1 maps in **Set**.

(b) All of the classes $\mathcal{I}, \mathcal{R}, \mathcal{L} \subset$ **Pos** in Examples 4.21 satisfy (a), so define preorders.

(c) Of these, $\mathcal{R}$ satisfies part (b), taken on its own, *i.e.* when A is an $\mathcal{R}$-ensemble but D is an arbitrary coalgebra.

(d) The classes $\mathcal{I}$ and $\mathcal{L}$ do not satisfy (b) on its own.

(e) If $\alpha \in \mathcal{L}$ and $\delta \in \mathcal{I}$ (plain mono) then any homomorphism $A \longrightarrow D$ belongs to $\mathcal{L}$.

(f) Vertical and prone maps for a fibration form a factorisation system, but the prone maps need not satisfy the stronger cancellation property because there can be vertical split idempotents. □

The preorder **Ens** inherits the ipo structure from the underlying category:

Lemma 7.5 The forgetful functors

$$\mathbf{Ens} \longrightarrow \mathbf{WfCoAlg} \longrightarrow \mathbf{CoAlg} \longrightarrow \mathcal{C}$$

create filtered colimits and the initial object $\emptyset$.

Proof For **WfCoAlg** $\to \mathcal{C}$ this is Proposition 5.10. For **Ens** we use the requirements on $\emptyset$ and directed unions in Assumption 4.19. □

Lemma 7.6 The functor T preserves ensembles. The only one that it fixes (up to isomorphism) is the initial algebra, if this exists.

Proof Since T preserves well-foundedness by Lemma 5.1 and $\mathcal{M}$ by assumption. The second part is Corollary 6.11. □

Theorem 7.7 The functor T has an initial algebra iff it has a set of isomorphism classes of ensembles.

Proof If there is an initial algebra then it is an ensemble by Propositions 6.9 and 6.10 and other ensembles are initial segments of it by Lemma 7.3. Being well powered (Definition 4.7) says that there is a "set" of isomorphism classes of these.

The word "set" in the converse must therefore be understood in the same way: it means that the preorder **Ens** is equivalent to an *internal* poset in $\mathcal{S}$. This is an ipo by Lemma 7.5 and the functor T provides its successor operation, to which we apply Pataraia's Theorem 2.4. The unique fixed point of T is the initial algebra. □

This set, or its cardinality, is known as the ***rank*** of the functor T, although the word rank is also used for the ordinal reflection of any well founded relation.

Applying Theorem 5.12 to this, we deduce

Corollary 7.8 If there is an initial algebra, it satisfies any property of coalgebras that holds of the initial object of $\mathcal{C}$ and is preserved by isomorphism, the functor and filtered colimits. □

Remark 7.9 Now we will show that the preorder **Ens** has binary meets.

If this were a typical category-theoretic problem, we would approach it by lifting pullbacks from the underlying category to coalgebras and ensembles. However, we should not expect to be able to do that unless the functor T preserves intersections (Proposition 9.5).

Even if we could construct pullbacks easily, we would still have to choose *roots* for them, *i.e.* the target corners of the squares. In a preorder these are just called common upper bounds, the least of which is the *join*. However, as we will see in Section 10 and [Tay25c], constructing this is a difficult problem in itself and depends on finding the meet first. Also, the terminal (or greatest extensional) well founded coalgebra would be the initial algebra, which need not exist.

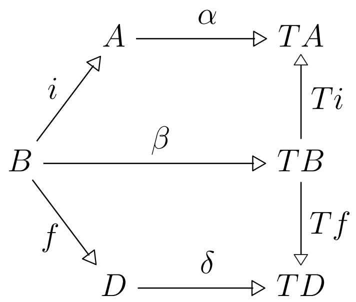
What we have instead, and without that assumption, is a generalisation of *set-theoretic* intersection. Indeed, the idea goes back to Cantor's original investigation of the classical ordinals, which he "zipped together" [Can97, §13 Thms N&E].

The diagram in the main Construction may be daunting, but it is just a double version of Construction 5.2 for subcoalgebras. Alternatively, it is an adaptation of Construction 6.5 for attempts in the Recursion Theorem, where the target is now a *co*algebra, considered as a *partial* algebra whose evaluation part is the identity. It more complicated because we have to trim the support according to the partial target.

If you would like to compare this with the relational version in Proposition 2.12, recall from Lemma 3.3 that coalgebra homomorphisms define *bi*simulations. Since such relations are composable and reversible, the span of homomorphisms becomes a single relation.

Functionality for relations corresponds to i being mono and the partial bijection to both i and f being mono.

Definition 7.10 A ***span*** between two coalgebras (A, α) and (D, δ) is a pair (i, f) of coalgebra homomorphisms:

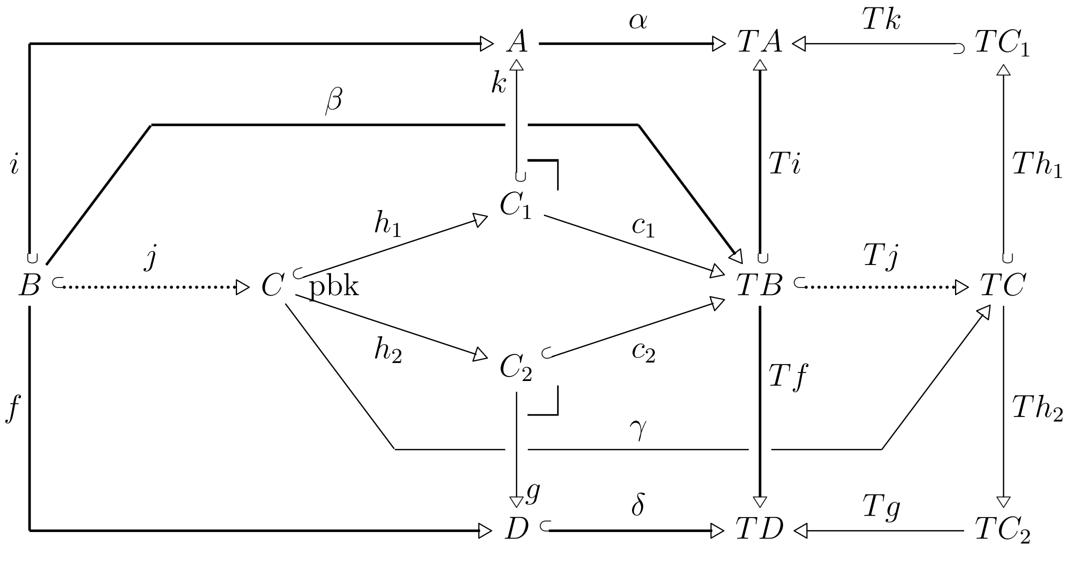


We will just use this in the case where all of the coalgebras are extensional and (B is) well founded. Hence, by Lemma 7.3, all of the homomorphisms are initial segments.

Beware, therefore, that we are relying heavily on the Assumptions 4.19 about the class of monos, as well as the foregoing theory of well-foundedness.

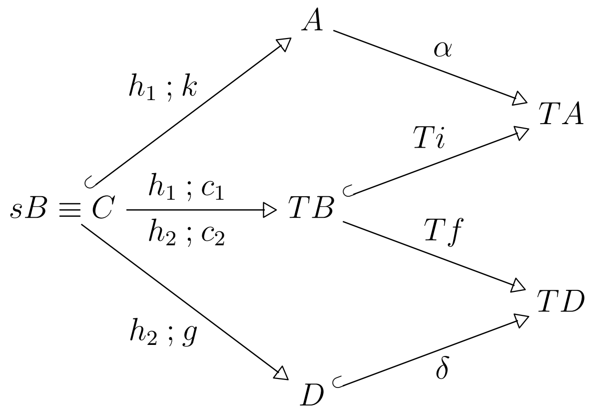
Construction 7.11 The ***relative successor*** of a span $(A, \alpha) \xleftarrow{i} (B, \beta) \xrightarrow{f} (D, \delta)$.

Proof When A, D and B are ensembles, all of the maps in the diagram are initial segments, by Lemma 7.3, whilst for general coalgebras they need not be. However, we just mark *some* of them as mono to indicate how this assumption for i and δ evolves in the construction. From this we see that it is not possible to separate the two roles of monos, *cf.* Notation 4.13. Well-foundedness is not assumed in this Construction itself.



Let C_1, C_2 and C be the pullbacks shown. If i and δ are initial segments then so are Ti, k, c_2, h_1 and $h_1 ; k$, because T, pullback and composition preserve them (Definition 4.15).

This construction makes C the limit of a W-diagram that is essentially a double form of the pullback in Construction 5.2:



Now B is the vertex of another cone over the W, with arrows i, β and f. Hence there is a unique mediator $j : B \to C$ to the limit, with

$$i \;=\; j\,;h_1\,;k, \qquad \beta \;=\; j\,;h_1\,;c_1 \;=\; j\,;h_2\,;c_2 \qquad \text{and} \qquad f \;=\; j\,;h_2\,;g.$$

When the previous maps, in particular i and $h_1 ; k$, are initial segments, so is j by the Cancellation Lemma 4.16.

Now we make C a coalgebra by defining $\gamma \equiv h_1 ; c_1 ; Tj$. Then j is a homomorphism because

$$j\,;\gamma \;\equiv\; j\,;h_1\,;c_1\,;Tj \;=\; j\,;h_2\,;c_2\,;Tj \;=\; \beta\,;Tj.$$

The new span with support C is given by $h_1 ; k : C \to A$ and $h_2 ; g : C \to D$, whose composites with j are i and f. Then $h_1 ; k$ and $h_2 ; g$ are homomorphisms because

$$(h_1\,;k)\,;\alpha \;=\; h_1\,;c_1\,;Ti \;=\; h_1\,;c_1\,;Tj\,;T(h_1\,;k) \;=\; \gamma\,;T(h_1\,;k)$$

and $$(h_2\,;g)\,;\delta \;=\; h_2\,;c_2\,;Tf \;=\; h_2\,;c_2\,;Tj\,;T(h_2\,;g) \;=\; \gamma\,;T(h_2\,;g),$$

where we use T applied to earlier equations. The remaining maps in the diagram may be shown to be initial segments by interchanging the roles of A and D. □

Lemma 7.12 The relative successor s is monotone (functorial) in $B' \longrightarrow B$, *cf.* Lemma 5.4.

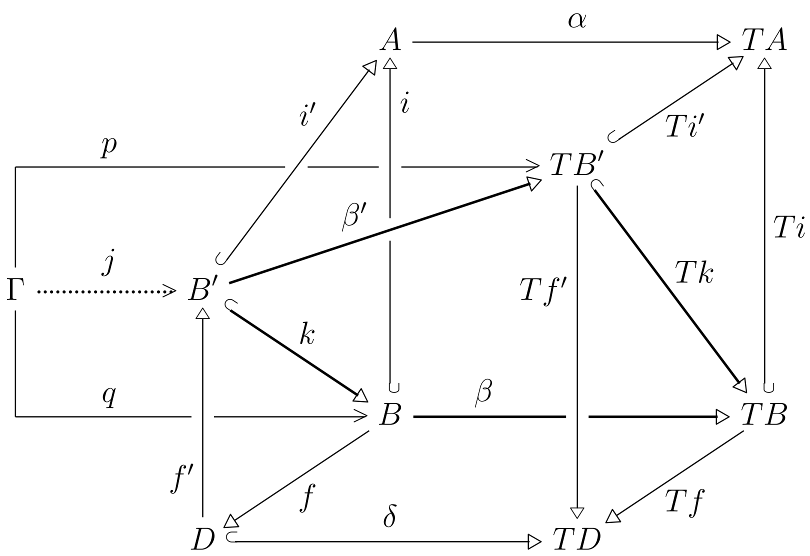
Proof The proof amounts to the mediator between two W-limits that share the nodes A, TA, D and TD but differ on $TB' \longrightarrow TB$, *cf.* the next diagram. The map $j : B \hookrightarrow C$ in the Construction makes the successor inflationary. □

Lemma 7.13 If B is well founded then so is $C \equiv sB$.

Proof By Lemma 5.8, since C is sandwiched between B and TB. □

The next result is maximality of fixed points for Pataraia's Theorem 2.4. It replaces the single pullback in Lemma 5.5 with a double one, the W-limit.

Lemma 7.14 If B is well founded and $B' \cong sB' \overset{k}{\hookrightarrow} B$ then $k : B' \cong B$.



Proof That k is a homomorphism of attempts means

$$i' = k\,;i \qquad \text{and} \qquad f' = k\,;f.$$

That B' is a fixed point means that it is already the limit of the W that defines its successor, namely

$$A \xrightarrow{\alpha} TA \xleftarrow{Ti'} TB' \xrightarrow{Tf'} TD \xleftarrow{\delta} D.$$

We claim that the homomorphism quadrilateral for $B' \overset{k}{\hookrightarrow} B$ (shown in bold) is a pullback, so let Γ be the vertex of a cone, with $p\,;Tk = q\,;\beta$. Then

$$q\,;i : \Gamma \to A, \qquad p : \Gamma \to TB' \qquad \text{and} \qquad q\,;f : \Gamma \to D$$

define a cone over the W for B' because

$$q\,;i\,;\alpha \;=\; q\,;\beta\,;Ti \;=\; p\,;Tk\,;Ti \;=\; p\,;Ti'$$

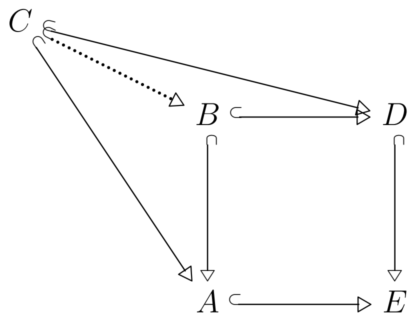
and $$q\,;f\,;\delta \;=\; q\,;\beta\,;Tf \;=\; p\,;Tk\,;Tf \;=\; p\,;Tf'.$$

Since B' is the limit, there is a unique mediator $j : \Gamma \to B'$ with

$$j\,;i' \;=\; j\,;k\,;i \;=\; q\,;i, \qquad j\,;\beta' \;=\; p \qquad \text{and} \qquad j\,;f' \;=\; q\,;f$$

whence $j\,;k = q$ since i is (plain) mono. Thus j provides the mediator that is required for B' to be the pullback. However, B is well founded by hypothesis, so any such pullback degenerates, making $k : B' \cong B$. □

Proposition 7.15 Ens is a preorder with binary meets. Moreover, whenever a meet-span B of ensembles is part of a commutative square of them then this is a pullback in **Ens**, but not necessarily in $\mathcal{C}$ or **CoAlg**.



Proof In any preorder considered as a category, the universal property of the meet of two objects is exactly that there is a greatest span between them.

Using Assumption 4.19, there is a unique span with $B \equiv \emptyset$. For any directed union, there is a unique pair of mediators and they are initial segments. As before, to make the category equivalent to an internal ipo, we need a well-poweredness condition, which comes from the requirement that (at least half of) the maps in Construction 7.11 are initial segments.

Therefore, the ipo of spans has a top element by Pataraia's Theorem 2.4.

There is no need for a common upper bound E to define a meet in a preorder, in the way that there is for a pullback in a *category*. However, if such E does exist then any pair $A \hookleftarrow C \hookrightarrow D$ of homomorphisms from an ensemble form a commuting square, since there is at most one homomorphism $C \longrightarrow E$ by Lemma 7.3(a).

Therefore, since E played no active role in this argument, the universal property of the pullback holds irrespective of its root. □

Switching back from preorders to categorical language, the meet therefore has the property that we would normally call a *product*. We avoid that word because the construction looks like set-theoretic *intersection* and nothing like the *Cartesian* product or its encoding in set theory. Set-theoretically, the maps $A \leftarrow C \to D$ do have to be subset inclusions and not arbitrary functions. Categorically, $A \leftarrow C \to D$ must be coalgebra monomorphisms and not just $\mathcal{C}$-maps.

Also, we have only allowed C to be another *extensional* coalgebra, not a general well founded one. It should be possible to prove this using a yet more complicated version of Construction 7.11, but it will be easier with the extra structure of the next section and we will do that in [Tay25a].

Examples 7.16 The consequences for the classes $\mathcal{I}, \mathcal{R}, \mathcal{L} \subset$ **Pos** in Example 4.21 are that:

(a) for $\mathcal{I}$, there are homomorphisms between these "extensional" well founded $\mathcal{D}$-coalgebras that are not necessarily mono;

(b) for $\mathcal{R}$, meets like this exist but may be different from the pullbacks in **Pos**;

(c) for $\mathcal{L}$, the two constructions for meets agree, but we have to use *this* one as a step towards finding the common upper bounds that make the pullback meaningful. □

To sum up this section, in particular Corollary 6.11, Lemmas 7.3–7.6 and Proposition 7.15:

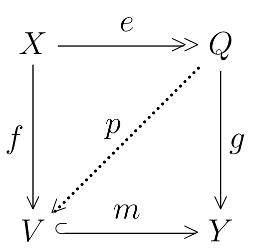
Theorem 7.17 The category **Ens** of ensembles and coalgebra homomorphisms
(a) is a preorder with
(b) a least object,
(c) directed unions,
(d) binary meets and
(e) an inflationary monotone successor, namely the functor T.
Moreover,
(f) the greatest (terminal) ensemble is the initial algebra, if either of these exists, and is the unique fixed point of T.
All of these statements about the preorder are to be understood "up to isomorphism".

8 Imposing the properties

In this section we show how to make a given coalgebra well founded and extensional:
(a) the greatest well founded part of a coalgebra provides the right adjoints to **WfCoAlg** $\to$ **CoAlg** and **Ens** $\to$ **ExtCoAlg**; and
(b) there is a "Mostowski extensional quotient" giving a left adjoint to the forgetful functor **Ens** $\to$ **WfCoAlg**.
In future work we will use similar methods
(c) to construct the "rank" according to various notions of ordinal; and
(d) to propose a categorical principle that is intended to serve the purpose of the axiom-scheme of replacement.

The key idea in doing these things is (the categorical abstraction of) the fact that any function can be expressed as the composite of a surjection and the inclusion of its image. One of the earliest achievements of category theory, or rather of Modern or Universal Algebra, was to bring together the various "isomorphism theorems" relating these for groups, rings, vector spaces, *etc.* The abstract formulation was given by Peter Freyd and Max Kelly [FK72]:

Definition 8.1 Two maps $e : X \twoheadrightarrow Q$ and $m : V \hookrightarrow Y$ in any category are ***orthogonal***, written $e \perp m$, if, for any two maps f and g such that the square commutes, there is a unique morphism $p : Q \to V$ making the two triangles commute:



Then a ***factorisation system*** is a pair of classes of morphisms $(\mathcal{E}, \mathcal{M})$ such that
(a) the classes $\mathcal{E}$ and $\mathcal{M}$ each contain all isomorphisms;
(b) they are each closed under composition;
(c) $e \perp m$ for every $e \in \mathcal{E}$ and $m \in \mathcal{M}$ and
(d) every morphism $f : X \to Y$ can be expressed as $f = e\,;m$ with $e \in \mathcal{E}$ and $m \in \mathcal{M}$.

Examples 8.2

(a) Inclusions (1–1 maps, monomorphisms) and surjections (onto maps, epimorphisms) in **Set** or a topos, where surjections are quotients by equivalence relations and this class is stable under pullbacks.

(b) More generally in type theories, if the factorisation is stable under pullback then the $\mathcal{E}$ class is associated with an existential quantifier [HP89] [Tay99, §9.3].

(c) In a general category with inverse images, a map e that is orthogonal to all monos is called an ***extremal epi*** and is characterised by $\forall m \in \mathcal{M}.\, e = m\,;f \Longrightarrow m = \mathsf{id}$.

(d) The class $\mathcal{I} \subset$ **Pos** of plain monos (Example 4.21) is part of a factorisation system, whose "epis" are surjective functions that generate the order on the codomain. However, the lowersets functor $\mathcal{D}$ does not preserve the class $\mathcal{I}$.

(e) The class $\mathcal{R}$ of regular monos in **Pos** is part of a factorisation system with plain epis, which are monotone functions that are surjective on points.

(f) The class $\mathcal{L}$ of lower inclusions is part of a factorisation system whose "epis" are cofinal functions in the usual sense, so they *do not* have the cancellation property and *are not* well co-powered.

(g) The vertical maps of any fibration (those that it takes to id) form the $\mathcal{E}$ class of a factorisation system, whose $\mathcal{M}$-maps are called prone, horizontal or cartesian; with slight modification, this may be used to provide a fibred account of this paper, *cf.* Definition 4.7.

(h) For any category $\mathcal{C}$ with pullbacks, in the arrow category $\mathcal{C}^{\rightarrow}$ the pullback squares form an $\mathcal{M}$-class of a factorisation system and the squares whose target arrow is invertible form the orthogonal $\mathcal{E}$-class.

On first reading this section you should just take everything according to Example (a).

There are many textbook accounts of factorisation systems, such as [Tay99, §5.7], but here are the facts that we shall need in particular:

Lemma 8.3 In a factorisation system,

(a) the factorisation $f = e\,; m$ is unique up to unique isomorphism;

(b) if $f \in \mathcal{E} \cap \mathcal{M}$ then f is an isomorphism;

(c) if the pullback of an $\mathcal{M}$-map exists in the category then it is also in $\mathcal{M}$ (so Assumptions 4.1(a) and 4.15(e) become redundant);

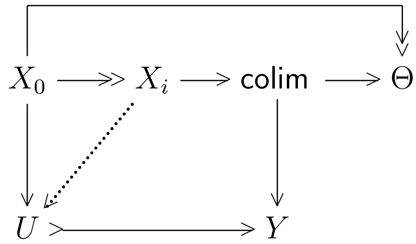
(d) $\mathcal{E}$ has the cancellation property that if $f,\ (f\,; e) \in \mathcal{E}$ then $e \in \mathcal{E}$, *cf.* Lemma 4.16;

(e) if $f \in \mathcal{C}$ has $f \perp m$ for all $m \in \mathcal{M}$ then $f \in \mathcal{E}$ (in fact quantification over $\mathcal{M}$ is not needed: we only use this for the $\mathcal{M}$-part of the factorisation of f);

(f) if $e = f\,; m$ with $e \in \mathcal{E}$ and $m \in \mathcal{M}$ then m is an isomorphism;

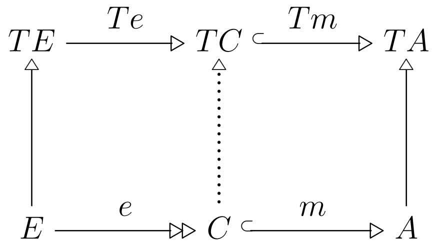
(g) if the maps in a directed or pushout diagram are all in $\mathcal{E}$ then so are those in the colimiting cocone; and

(h) the mediator from such a colimit to a cocone consisting of $\mathcal{E}$-maps is also in $\mathcal{E}$.



Proof (a–f) Easy, but see *e.g.* Lemma 5.7.6 and Proposition 5.7.7 in [Tay99]. (g) Any pushout has a *root* X_0 (with maps to all of the other vertices of the diagram), and any directed diagram is equivalent to one with a root. Using $(X_0 \twoheadrightarrow X_i) \perp (U \rightarrowtail Y)$, there is a unique mediator $X_i \to U$. These maps form a cocone, with mediator $\mathsf{colim} \to U$. Finally, (h) follows from (e,g). $\square$

Lemma 8.4 If the functor T preserves $\mathcal{M}$ then factorisation lifts to coalgebra homomorphisms.



Proof Let $f : E \longrightarrow A$ be a coalgebra homomorphism that factorises as $f = e \,;m$ in $\mathcal{C}$ with $e \in \mathcal{E}$ and $m \in \mathcal{M}$. Since T preserves $\mathcal{M}$ we have $e \perp Tm$ and so there is a unique map $C \to TC$ making the squares commute, *i.e.* factorising the homomorphism. $\square$

Indeed all of the factorisations that we do in this section are of homomorphisms. The notion could be expressed directly for them and it may be necessary to do this in future applications, not just to lift them from the underlying category $\mathcal{C}$.

Assumption 8.5 Instead of Assumption 4.19, we now require initial segments to form part of a factorisation system $(\mathcal{E}, \mathcal{M})$, whose $\mathcal{E}$-maps we call ***cofinal***.

As in Remark 4.20 for initial segments, this word is intended to hint at certain intuitions that are linked to the fact that all of the maps that we call cofinal are coalgebra homomorphisms. However, the notion is not necessarily the same as the traditional order-theoretic one. It is a novelty of this work that $\mathcal{E}$ can be a special class of morphisms in a category that need not be a topos.

We consider well-foundedness first.

Remark 8.6 Proposition 5.14 showed that any coalgebra A has a largest well founded subcoalgebra WA. Is W a functor and right adjoint to the inclusion **WfCoAlg** $\to$ **CoAlg**?

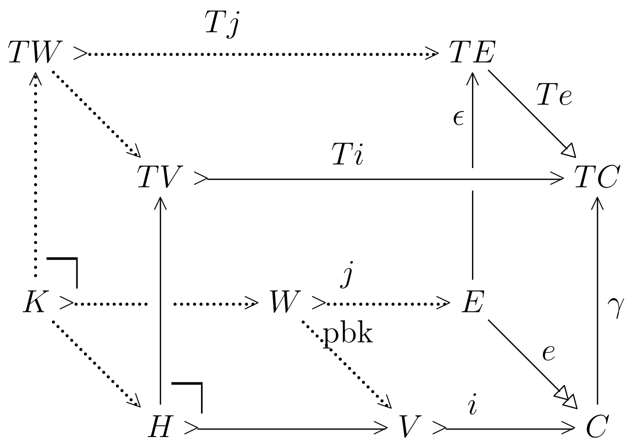
In order to show this, we require the predicates and initial segments to belong to the *same* class of monos. Recall that for Lemma 5.5 we already needed all initial segments to be predicates.

On the other hand, by Proposition 5.15, the subcoalgebra $\epsilon_A : WA \hookrightarrow A$ does not depend on which class of monos we used for the predicates to define well-foundedness.

Therefore we obtain the same result if we cut this class down to just the initial segments and use the same factorisation system for both purposes.

Then we have a categorical version of Corollary 2.15:

Lemma 8.7 Let E be a well founded coalgebra and $e : E \twoheadrightarrow C$ a cofinal homomorphism. Then C is also well founded.



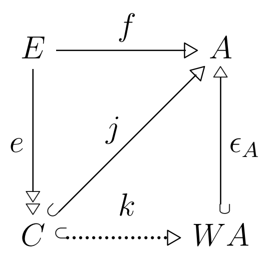
Proof Let $i : V \rightarrowtail C$ be a predicate that satisfies the induction premise given by the broken pullback from H to TC (at the front).

Pull this back along the homomorphism $e : E \twoheadrightarrow C$, in the sense of Lemma 5.6.

By well-foundedness of E, we have $j : W \cong E$.

Since $(e : E \twoheadrightarrow C) \in \mathcal{E}$ and it factors through $(i : V \rightarrowtail C) \in \mathcal{M}$, the latter is also an isomorphism by Lemma 8.3(f). $\square$

Theorem 8.8 The inclusion **WfCoAlg** $\rightarrowtail$ **CoAlg** has a right adjoint (coreflection), W, whose counit is an initial segment. This construction preserves extensionality, *i.e.* the well founded part of any extensional coalgebra is still extensional.



Proof We must show that any coalgebra homomorphism $f : E \longrightarrow A$ with E well founded factors uniquely through ϵ_A.

Let $E \xrightarrow{e} C \xhookrightarrow{j} A$ be the factorisation in $\mathcal{C}$ of f as a cofinal map followed by an initial segment. By Lemma 8.4, C is a coalgebra and e and j are homomorphisms. Then C is well founded by Lemma 8.7 and it is a subcoalgebra of A by construction.

It is therefore a subcoalgebra of WA, since WA was the largest such. The map $E \longrightarrow WA$ is unique since $\epsilon_A : WA \hookrightarrow A$ is mono.

If A is extensional then so is WA by Lemma 7.2. $\square$

This gives a simpler proof of Proposition 5.10, that the forgetful functor **WfCoAlg** $\longrightarrow$ **CoAlg** creates colimits.

Now we turn to the imposition of extensionality, for which the Mostowski quotient is the motivating example (Remark 1.11). There are different situations, depending on whether we have control over $\mathcal{M}$ or $\mathcal{E}$. If $\mathcal{E}$ is too large then we must consider Replacement [Tay25a]. The assumption that T preserves $\mathcal{M}$ can also be dropped, but that makes things more complicated [Tay25c].

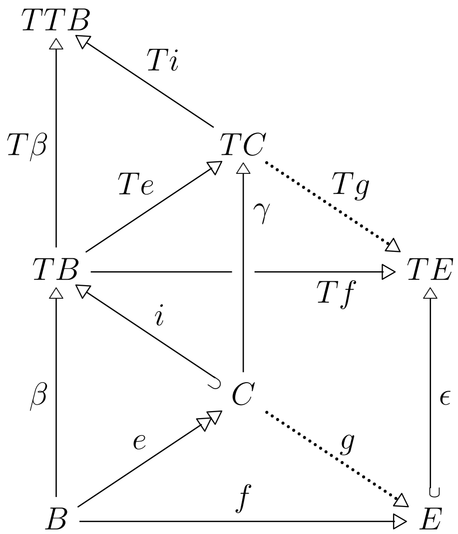
In the familiar setting, this quotient construction provides an onto function between sets with binary relations, so it corresponds to an equivalence relation, which we constructed in Theorem 2.17. However, as we have seen throughout this work, iterative constructions are often better understood by making a more detailed study of the single *step* and then using a general-purpose tool for the iteration. In this case, the step is the *factorisation* of the structure map into its onto and 1–1 parts. That is what we do now, using our more abstract categorical tools; it requires fewer assumptions than one might suppose.

Construction 8.9 The ***successor quotient*** (C, γ) of any coalgebra (B, β) is given by factorising β as a cofinal homomorphism followed by an initial segment, as shown below.

If B is well founded then so is C.

B is fixed by the construction ($e : B \cong C$) iff it is extensional ($\beta \in \mathcal{M}$).

Any homomorphism $f : (B, \beta) \longrightarrow (E, \epsilon)$ to an extensional coalgebra factors uniquely through C.



Proof Let $\beta = e\,;i$ be the factorisation, *via* C, and put $\gamma \equiv i\,;Te$. Then the three triangles on the left commute, so (C, γ) is a coalgebra and $e : B \twoheadrightarrow C$ and $i : C \hookrightarrow TB$ are homomorphisms.

The factorisation preserves well-foundedness, by Lemma 5.8 or 8.7.

If $e : B \cong C$ then $\beta \cong i \in \mathcal{M}$, so B is extensional, and conversely.

Since $e : B \twoheadrightarrow C$ is orthogonal to $\epsilon : E \hookrightarrow TE$, there is a unique map $g : C \to E$ such that $e\,;g = f$ and $i\,;Tf = g\,;\epsilon$. Hence g is a homomorphism:

$$\gamma\,;Tg \;\equiv\; i\,;Te\,;Tg \;=\; i\,;Tf \;=\; g\,;\epsilon. \qquad \square$$

Assumptions 8.10

(a) Cofinal maps (the class $\mathcal{E}$) have the cancellation property for epis, $e\,;f = e\,;g \Rightarrow f = g$;

(b) this class is well co-powered; and

(c) initial segments have the strong cancellation property $h\,;\delta \in \mathcal{M} \Rightarrow h \in \mathcal{M}$ that was needed for Lemma 7.3(b).

Lemma 8.11 Each co-slice category $X/\mathcal{E}$ (whose objects are $\mathcal{E}$-maps *from* X and whose morphisms are commutative triangles) is equivalent to an ipo. This holds in $\mathcal{C}$, **CoAlg** and **WfCoAlg**.

Proof By the cancellation property for epis the co-slice category is a preorder.

The notion of being well *co*powered in Assumption 8.10 is the obvious analogue of being well powered (Definition 4.7), *i.e.* that the external category is equivalent to an internal poset in the base topos $\mathcal{S}$.

The identity id_X is its initial or least object and filtered colimits provide directed joins.

The same argument restricts from $\mathcal{C}$ to **CoAlg** and **WfCoAlg** because colimits are created (Proposition 5.10). By Lemma 8.7, if X is well founded then so too is everything in $X/\mathcal{E}$. □

Lemma 8.12 Any well founded coalgebra A has a longest outgoing cofinal homomorphism and its target is extensional.

Proof By this we mean that the ipo has a greatest element, for which we prove maximality of fixed points in Theorem 2.4 for the inflationary monotone endofunction that is defined by Construction 8.9.

Suppose that, in the ipo, $(B,\beta) \le (C,\gamma)$ with B fixed. This means that (B,β) is extensional and there is a cofinal ($\mathcal{E}$) homomorphism $B \twoheadrightarrow C$. However, by Lemma 7.3(b), any such homomorphism is an initial segment (in $\mathcal{M}$) and is therefore an isomorphism. □

Theorem 8.13 The forgetful functor **Ens** → **WfCoAlg** has a left adjoint, whose unit $\eta_A : A \twoheadrightarrow RA$ is cofinal.

Proof Let $\eta_A : A \twoheadrightarrow C \equiv RA$ be the longest $\mathcal{E}$-map. Being fixed, C is extensional.

For the universal property, let $A \longrightarrow E$ be a homomorphism to an extensional coalgebra. This factors uniquely through the successor quotient by Construction 8.9, and through colimits by definition. Hence by Pataraia induction it does so through the least fixed point, which therefore yields the left adjoint. □

Corollary 8.14 If $\mathcal{C}$ has pushouts then so does **Ens**.

Proof The forgetful functor **WfCoAlg** → **CoAlg** → $\mathcal{C}$ create colimits (Proposition 5.10) and any left adjoint preserves them. That is, the pushout in $\mathcal{C}$ gives that in **WfCoAlg** and then we apply the extensional reflection to yield the pushout in **Ens**. □

Remark 8.15 The ideas in this section were motivated by the properties of onto and 1–1 functions between sets. That category also has "nice" pushouts (*disjoint* unions) that we will discuss in Section 10. However in order to understand the intricacies of using general factorisation systems, we should look at the three of them in **Pos** (Examples 8.2(d–f)). They are studied fully in [Tay25c].

(a) Everything in this section works for $\mathcal{R}$. However, although the Corollary provides pushouts of well founded $\mathcal{R}$-extensional coalgebras , they are not as nice that those for sets. They have additional instances of the order relation that do not come from either component, with the result that some pairs of elements become identified. (This is the typical effect of obtaining colimits from reflection functors.)

(b) For $\mathcal{I}$, the notion of well-foundedness is meaningless because the functor $T \equiv \mathcal{D}$ doesn't preserve this class, whilst almost everything in this paper depends on that . However, Construction 8.9 and Lemma 8.11 do survive without that Assumption, as does Theorem 8.13 with some modification, giving an extensional reflection without well-foundedness.

(c) Pushouts of $\mathcal{L}$-extensional coalgebras in **Pos** are nice like those for sets, but for entirely different reasons.

Remark 8.16 More seriously, the "cofinal" maps orthogonal to the class $\mathcal{L}$ do not obey Assumption 8.10 at all, because they are far from being surjective on points. Then, not only does the Theorem fail, but we cannot even *form* the family of objects in the ipo within the logic of an elementary topos or *Zermelo* set theory. We need the Axiom-Scheme of Replacement, or rather a natural categorical axiom to replace it [Tay25a].

9 When the functor preserves inverse images

Previous work on this subject required the functor to preserve inverse images of monos, but this new account has only used preservation of the monos themselves. We now re-impose the stronger assumption and prove the relatively few earlier results that depend on it. Principal amongst these is Proposition 1.4, which is a very important result for the way that well founded relations are used in practice across mathematics.

There is a second essential requirement for this proof, namely the *universal quantifier*. In the categorical formulation this appears in the form of the adjunction $f^* \dashv f_*$. Gerhard Osius noted this in his version of the result [Osi74, Prop 6.3(a)]. Any topos has it (Notation 3.2), but since we are considering more general categories, we state it as a further assumption on the subobjects:

Assumption 9.1 In addition to the assumptions in Section 4,

(a) the functor $T : \mathcal{C} \to \mathcal{C}$ must preserve inverse image diagrams of predicates along coalgebra homomorphisms ; and

(b) each inverse image operation f^* must have a right adjoint f_* on predicates, at least when f is a coalgebra homomorphism .

Besides 1–1 maps in **Set**, the results in this section also hold for $\mathcal{L} \subset$ **Pos** but not $\mathcal{I}$ or $\mathcal{R}$ in Example 4.21 [Tay25c].

As an aid to understanding our categorical proof, we first give it for well founded *relations*, in as similar a form as possible. See [Tay99, Prop 2.6.2] for a box-style proof in natural deduction for well founded relations.

Proposition 9.2 Let $(A, \prec)$ be a well founded relation and $f : (B, <) \to (A, \prec)$ a ***strictly monotone*** function in the sense that

$$\forall bb' {:} B. \quad b' < b \implies fb' \prec fb$$

then $(B, <)$ is also well founded.

Proof Let ψ be a predicate on B satisfying the induction premise

$$\forall b. \qquad (\forall b'.\, b' < b \Rightarrow \psi b') \Longrightarrow \psi b.$$

For comparison with the categorical proof below, *cf.* Proposition 3.6, this is $K \subset V$, where

$$K \;\equiv\; \{b : B \mid \forall b'.\, b' < b \Rightarrow \psi b'\} \subset B \qquad \text{and} \qquad V \;\equiv\; \{b : B \mid \psi b\} \subset B.$$

The key step is to define $f_* V \equiv \{a : A \mid \phi a\} \subset A$, where $\phi a \;\equiv\; (\forall b'.\, fb' = a \Rightarrow \psi b)$, and

$$H \;\equiv\; \{a : A \mid \forall a'.\, a' \prec a \Rightarrow \phi a'\} \;\equiv\; \{a : A \mid \forall b'.\, fb' \prec a \Longrightarrow \psi b'\} \;\subset\; A.$$

Strict monotonicity and the induction premise give $f^* H \subset K \subset V$, which is

$$\forall b. \qquad (\forall b'.\, fb' \prec fb \Rightarrow \psi b') \Longrightarrow (\forall b'.\, b' < b \Rightarrow \psi b') \Longrightarrow \psi b.$$

Quantifying over $\{b' \mid fb' = a\}$, we obtain $H \subset f_* V$, which is

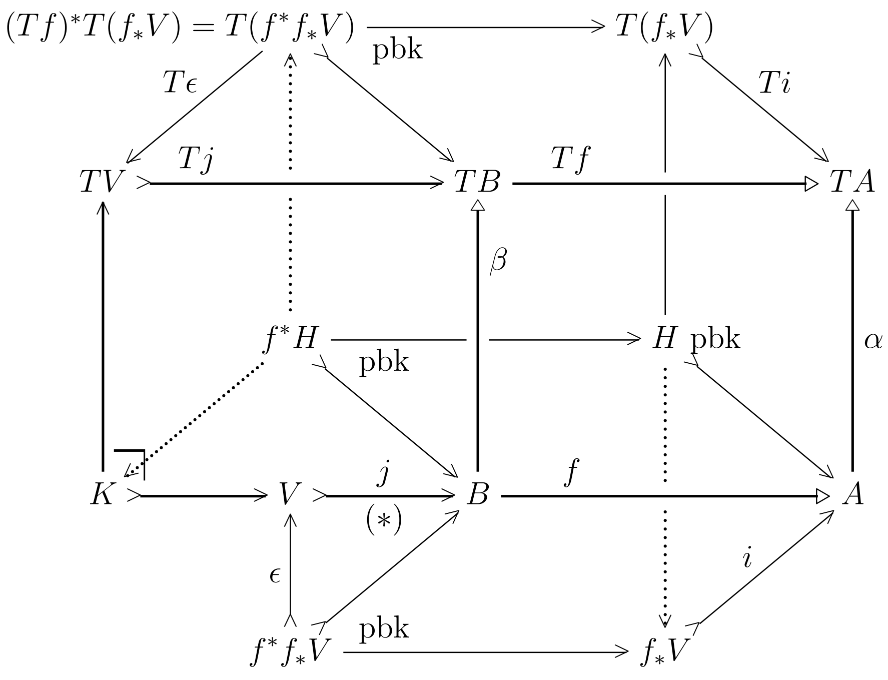
$$\forall a. \qquad (\forall a'.\, a' \prec a \Rightarrow \phi a') \iff (\forall b'.\, fb' \prec a \Rightarrow \psi b') \Longrightarrow (\forall b'.\, fb' = a \Rightarrow \psi b') \;\equiv\; \phi a.$$

Then $\forall a.\, \phi a$ since $(A, \prec)$ is well founded, whence $\forall b.\, \psi b$ as required. $\square$

We now prove the result for general functors that preserve inverse images and coalgebra homomorphisms that are equipped with f_*. Notice, however, that the hypothesis that f be a coalgebra homomorphism is actually stronger (in the case of $T \equiv \mathcal{P}$) than being strictly monotone (*cf.* Lemma 3.3).

Theorem 9.3 Let $f : (B, \beta) \longrightarrow (A, \alpha)$ be a coalgebra homomorphism with f_*, where (A, α) is well founded. Then (B, β) is also well founded.

Proof Given the diagram marked in thick lines, apply the right adjoint f_* to $j : V \rightarrowtail B$, to get $i : f_* V \rightarrowtail A$. The counit of this adjunction is $\epsilon : f^* f_* V \to V$ and makes the little triangle $(*)$ commute, where f^* is given by pullback (inverse image) of i along f. The upper part of the diagram is the T-image of the lower part, including this pullback but not K. Let $H \equiv \alpha^* T(f_* V)$ be the pullback of Ti and α and $f^* H$ its pullback along f.



By construction, the whole diagram of solid lines commutes from f^*H to TA. In particular, $f^*H \rightarrowtail B \to TB$ and $f^*H \to H \to T(f_*V)$ agree at TA, so there is a pullback mediator $f^*H \to T(f^*f_*V)$. Then $f^*H \to T(f^*f_*V) \to TV$ agrees with $f^*H \rightarrowtail B$ at TB, so there is also a pullback mediator $f^*H \to K$.

This shows that $f^*H \subset K \subset V$ as $\mathcal{C}$-subobjects of B. Therefore, by the adjunction $f^* \dashv f_*$, we have $H \subset f_*V$ as subobjects of A.

That is, there is a map $H \to f_*V$ that makes the right-hand part of the diagram into a broken pullback. Now, since A is well founded, $i : f_*V \cong A$, so $f^*f_*V \cong B$ and $j : V \cong B$ [Tay96b, Prop 7.3]. □

Examples 9.4 To show that the additional hypotheses are necessary, we substitute preorders for categories in the whole theory, so a well founded *coalgebra* becomes a well founded *element* in the sense of Lemma 2.3.

$$\begin{array}{ccc} y & \leq & sy = ss\bot \\ \text{\rotatebox{90}{$\leq$}} & & \text{\rotatebox{90}{$\leq$}} \\ \bot & \leq & s\bot \end{array} \qquad \begin{array}{ccccccccccc} y & \leq & sy & \leq & ssy & \leq & sssy & \leq & \cdots & \leq & s^\infty y \\ \text{\rotatebox{90}{$\leq$}} & & \text{\rotatebox{90}{$\leq$}} & & \text{\rotatebox{90}{$\leq$}} & & \text{\rotatebox{90}{$\leq$}} & & & & \| \\ \bot & \leq & s\bot & \leq & ss\bot & \leq & sss\bot & \leq & \cdots & \leq & s^\infty\bot \end{array}$$

In both diagrams, the elements $s^n\bot$ and $s^\infty\bot$ are well founded because

$$\forall x. \quad s(s^n\bot) \wedge x \leq (s^n\bot) \Longrightarrow x \leq (s^n\bot),$$

but y is not, because $s\bot \wedge y \leq \bot$ but $y \not\leq \bot$. Tight well-foundedness behaves in the same way.

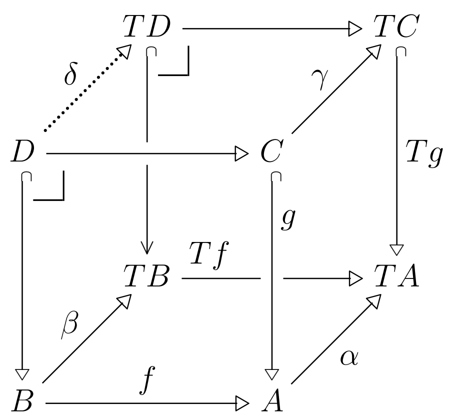
The first example is a Heyting semilattice, but s does not preserve the meet $y \wedge s\bot = \bot$.

The second is also distributive but it is not a Heyting semilattice, since $y \wedge (-)$ does not preserve the directed join $\bigvee^{\uparrow} s^n\bot$. However, s preserves meets because, for $n < \infty$ and $m \leq \infty$,

$$s^n\bot \;\wedge\; s^m y \quad = \quad s^{\mathsf{min}(n,m)}\bot. \qquad \square$$

The Theorem seems to be needed to construct inverse images of well founded coalgebras.

Proposition 9.5 The functors **Ens** $\longrightarrow$ **WfCoAlg** $\longrightarrow$ **CoAlg** $\longrightarrow$ $\mathcal{C}$ create inverse images.



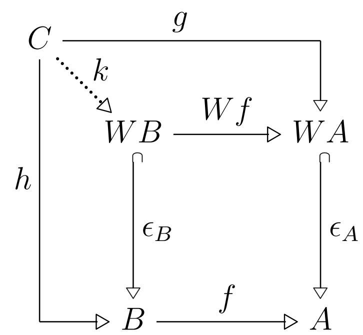
Proof The diagram shows how to find the inverse image (D, δ) of coalgebras $B \rightarrowtriangle A \leftarrowtriangle C$ for a functor T that preserves them. The structure map δ is the mediator to the pullback with T applied.

If either of the given coalgebras B or C is well founded then so is D, by Theorem 9.3. If B is extensional then so is D by Lemma 7.2. □

Recall that Corollary 7.15 gave a quite different construction of binary meets, which are pullbacks in **Ens** if common upper bounds exist. However, if T does not preserve inverse images in $\mathcal{C}$ then the pullbacks in **Ens** and $\mathcal{C}$ need not agree [Tay25c].

Now that we have a straightforward way of working with pullbacks of coalgebras, there are some other things we need to know about them:

Lemma 9.6 The naturality square for the counit $\epsilon : W \hookrightarrow \mathsf{id}$ of the well founded part functor is a pullback.



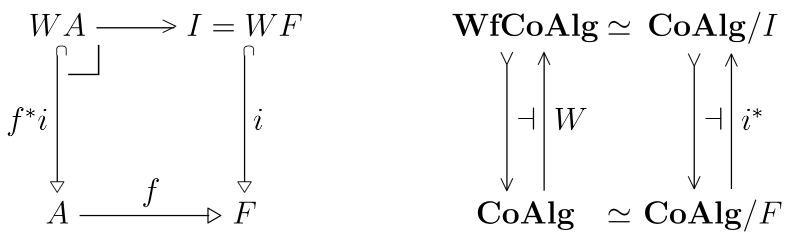
Proof Since WA is well founded, so is C by Theorem 9.3. Therefore $h : C \longrightarrow B$ factors through WB and k is unique such that $h = k \,;\epsilon_B$. Then

$$k\,; Wf\,; \epsilon_A \;=\; k\,; \epsilon_B\,; f \;=\; h\,; f \;=\; g\,; \epsilon_A,$$

so $k\,; Wf = g$ since ϵ_A is (plain) mono. □

Using this, we may construct the well founded part of any T-coalgebra (Theorem 8.8) in a uniform way:

Corollary 9.7 If T has a final coalgebra F (and hence an initial algebra I by Corollary 6.13) then the well founded part WA of any coalgebra A is given by the inverse image on the left:



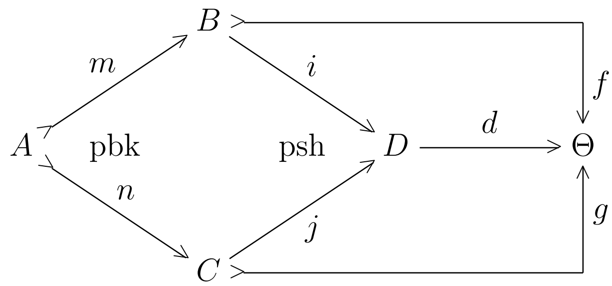
Proof Any category with a terminal object is equivalent to the slice by it. By Proposition 6.10, the initial algebra I is the terminal well founded coalgebra, whilst by Theorem 9.3, any coalgebra having a homomorphism to I is well founded. Hence we have equivalences as shown on the right, commuting with the forgetful functors. The latter both have right adjoints, which must also be equivalent. □

10 Pushouts

Besides the preservation of pullbacks, we have also avoided using pushouts (binary joins) in this work, by developing a much more delicate proof for *directed* joins that uses Pataraia's fixed point theorem. We now restore the pushouts, inspired by Gerhard Osius's treatment of them in his reconstruction of Set Theory within a topos.

We will still pay attention to our analysis of special classes of monos in the base category $\mathcal{C}$, but now the additional assumptions make it much more like **Set** or a topos than we have so far needed. Of the classes of "monos" in **Pos** (Example 4.21), $\mathcal{L}$ satisfies them but $\mathcal{I}$ and $\mathcal{R}$ do not [Tay25c].

In addition to the Assumptions that we have accumulated, we need a "union" property for pushouts analogous to Proposition 4.4 for directed unions. See [Bar87, LS05, GL12] for further accounts of well behaved pushouts.



We will discuss these properties of pushouts in **Set** and $\mathbf{Set}^{\mathsf{op}}$. The first result is known as the ***Amalgamation Lemma***:

Lemma 10.1 In **Set** or any pretopos, the pushout of a pair of monos $B \overset{m}{\leftarrowtail} A \overset{n}{\rightarrowtail} C$ is another pair of monos and is also a pullback.

Proof The following is a congruence:

$$(A+B)+(A+C) \underset{[m;\nu_1,\ \nu_0,\ n;\nu_0,\ \nu_1]}{\overset{[m;\nu_0,\ \nu_0,\ n;\nu_1,\ \nu_1]}{\rightrightarrows}} B+C.$$

If $f : B \to \Theta$ and $g : C \to \Theta$ make a commutative square then $[f,g] : B+C \to \Theta$ coequalises the congruence. Since the quotient is effective, to verify monos and equalisers, it suffices to inspect the congruence [FS90, 1.651] [Tay99, 5.8.10]. □

Lemma 10.2 The dual property also holds in **Set** or any effective regular category, such as a category of finitary algebras.

Proof The pullback of $B \overset{m}{\twoheadrightarrow} A \overset{n}{\twoheadleftarrow} C$ is $D \equiv \{(b,c) \mid mb = nc : A\}$. Suppose $B \overset{u}{\leftarrow} E \overset{v}{\to} C$ make a commutative square from D. For each $a : A$, since m and n are surjective there are $b : B$ and $c : C$ with $a = mb = nc$, so $(b,c) \in D$ and $ub = vc$. Then if $a = mb' = nc'$ too, also $ub = vc' = ub' = vc$. Hence we may unambiguously define the mediator $e : A \to E$ by $ea \equiv ub$. □

Lemma 10.3 In **Set** or any pretopos, if A, B, Θ and C form a pullback and A, B, D and C form a pushout, with all of these maps mono, then the mediator $d : D \to \Theta$ is also mono.

Proof Regarding pullbacks, first note that if the square rooted at D is one then so is that to Θ, but the converse requires $D \to \Theta$ to be (plain) mono.

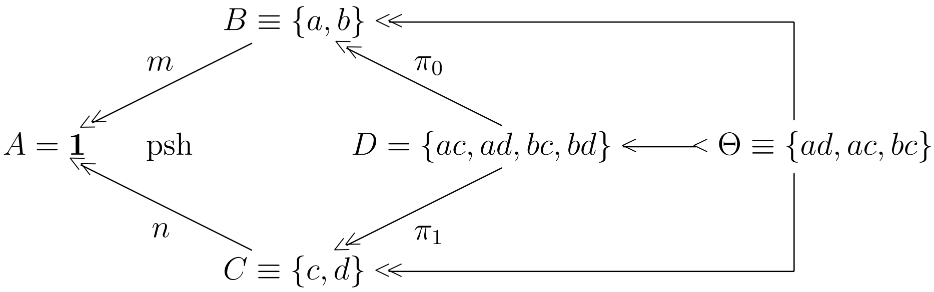
We consider the ***kernel*** of d (the pullback of d against itself), $K \subset D \times D$.

Since D is the union of its subobjects B and C and the pullback d^* preserves unions, $D \times D$ is the union of four parts, $B \times B$, $B \times C$, $C \times B$ and $C \times C$, and K is the union of their intersections with it. Putting these parts together, we have a surjection

$$\left.\begin{array}{lcccl} \mathsf{ker}\,(i\,;d) & = & K \cap B \times B & = & \Delta_B \\ \mathsf{pbk}\,(i\,;d,\ j\,;d) & = & K \cap B \times C & = & \Delta_A \\ \mathsf{pbk}\,(j\,;d,\ i\,;d) & = & K \cap C \times B & = & \Delta_A \\ \mathsf{ker}\,(j\,;d) & = & K \cap C \times C & = & \Delta_C \end{array}\right\} \longrightarrow\!\!\!\!\!\rightarrow K \ \subset\ D \times D$$

so the kernel $K \subset D \times D$ is $\Delta_B \cup \Delta_C$, which is the diagonal Δ_D. Hence d is mono, as required. □

Example 10.4 The dual of this Lemma fails in **Set**.



Proof Any pullback D rooted at $A \equiv \mathbf{1}$ is a product, so $D = B \times C$. For the whole diagram to commute $A \leftleftarrows \Theta$, using the same ideas as in Lemma 10.2, the three elements of Θ give the three equations

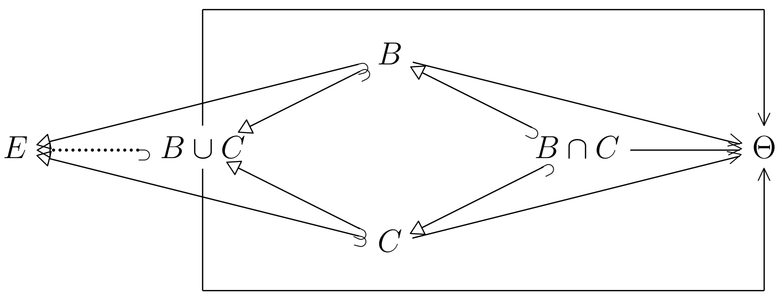
$$nd \ = \ ma \ = \ nc \ = \ mb \ : A,$$

whence the pushout rooted at Θ is $\mathbf{1}$. However, the pullback mediator $D \leftarrow \Theta$ is not epi. □

As in the previous section, the results in this one also hold for $\mathcal{L} \subset \mathbf{Pos}$ but not $\mathcal{I}$ or $\mathcal{R}$ in Example 4.21 [Tay25c].

Assumption 10.5 In addition to the assumptions of Sections 4 and 9 here we also require the underlying category $\mathcal{C}$ to have well-behaved pushouts of $\mathcal{M}$-maps in the foregoing sense.

Lemma 10.6 Well founded subcoalgebras and attempts admit binary joins.



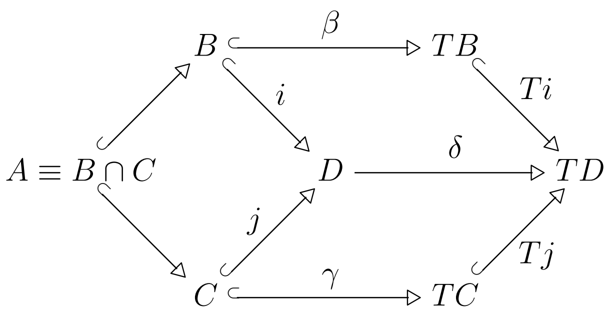
Proof Suppose that the outer diamond defines two attempts with well founded supports B and C. Let $B \cap C$ be the intersection (pullback) of these subobjects of E, so $B \cap C$ is a well founded coalgebra by Proposition 9.5 . Either by Lemma 6.4 using an equaliser or the alternative method following it that used Pataraia induction, the restrictions $B \cap C \to B \to \Theta$ and $B \cap C \to C \to \Theta$ agree. By the union property we have $A \hookleftarrow B \cup C \longrightarrow \Theta$. $\square$

We are now ready to give the categorical explanation of the strange "overlapping union" in Set Theory: putting B and C together does not yield a *coproduct* $B+C$ but their *pushout* rooted at their meet $A \equiv B \cap C$ from Corollary 7.15.

We already know from Proposition 5.10 that the functors **WfCoAlg** $\longrightarrow$ **CoAlg** $\longrightarrow \mathcal{C}$ create colimits. Also, by Theorem 7.17, **Ens** $\to$ **WfCoAlg** creates *filtered* colimits and the initial object. Corollary 8.14 showed that **Ens** has binary joins, but relied on the extensional reflection for this. What we show now is that they are inherited from $\mathcal{C}$, under the additional assumptions of this section.

The pushout is over the meet in **Ens**, given by Theorem 7.17, rather than that in Proposition 9.5. The reasons for this are explored further in [Tay25c].

Theorem 10.7 The preorder **Ens** has binary joins, given by pushout in $\mathcal{C}$ over the binary meet.



Proof Let (B, β) and (C, γ) be ensembles, so β and γ are initial segments.

By Theorem 7.17, they have a meet $A \equiv B \cap C$, and the maps $B \leftarrow A \to C$ are initial segments.

Let D be the pushout in $\mathcal{C}$; it is well founded by Proposition 5.10. By the union assumption (*cf.* Lemma 10.1), i and j are initial segments, as are Ti, Tj, $\beta\,;Ti$ and $\gamma\,;Tj$.

The key point is that $B \cap C$ is the pullback rooted at *either* D or TD, by Corollary 7.15.

Therefore $\delta : D \to TD$ is an initial segment by the union assumption (*cf.* Lemma 10.3), making D extensional. $\square$

The argument that Osius gave for this [Osi74, Thm 6.6] is rather more complicated (with a big diagram). Throughout his paper he used recursion instead of well-foundedness (*cf.* Proposition 3.10) and of course $T \equiv \mathcal{P}$, but for this particular result he used the partial map classifier $\tilde{C}$ (nowadays written $C_\perp$) in a topos.

We conclude by examining whether the functors that we have considered could possibly have further adjoints.

Question 10.8 Corollary 9.7 says that W is like inverse image, whilst Assumption 9.1(b) required that to have a right adjoint. Could W : **CoAlg** $\to$ **WfCoAlg** itself have a right adjoint? What would it mean?

Lemma 10.9 The forgetful functor **CoAlg** $\to \mathcal{C}$ has a right adjoint iff there is a final T-coalgebra in each slice $\mathcal{C}/X$. ☐

Lemma 10.10 If U : **CoAlg** $\to \mathcal{C}$ has a left adjoint L and there is a final coalgebra F then $F \cong \mathbf{1}$. Then the initial algebra and all ensembles are subobjects of $\mathbf{1}$.

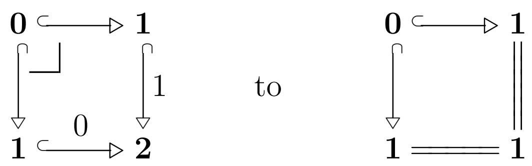
Proof For any $X \in \mathcal{C}$, there is a unique homomorphism $LX \longrightarrow F$ and so a unique $\mathcal{C}$-map $X \to F$, so $F \cong \mathbf{1}$. ☐

Lemma 10.11 If the forgetful functor U : **WfCoAlg** $\to$ **CoAlg** has left adjoint L then it and $U \dashv W$ are equivalences. If there is a final coalgebra then it would also be the initial algebra, *cf.* Peter Freyd's principle of ***algebraic compactness*** [Fre91].

Proof The unit $\eta : A \to ULA$ provides a map from *any* coalgebra to a well founded one, so by Proposition 9.3 A is well founded. ☐

Example 10.12 **Ens** $\to$ **WfCoAlg** does not create or even preserve colimits and so does not have a right adjoint: Binary coproducts are idempotent ($A + A \cong A$) in **Ens** but disjoint ($A \cap A = \emptyset$) (in $\mathcal{C}$ by hypothesis and so also) in **WfCoAlg**. For a concrete example, the extensional well founded relation $0 \prec 1$ is embedded twice in V. ☐

Example 10.13 The extensional reflection R : **WfCoAlg** $\to$ **Ens** does not have a left adjoint because it takes the pullback



where **2** carries the empty relation, which is well founded but not extensional.

Further work

This needs to be re-written.

The original purpose of this work was to provide an intuitionistic categorical account of transfinite recursion for my book [Tay99]. However, there was no way to use Hartogs' Lemma [Har15] constructively, and then out of the blue came Pataraia's far simpler proof of the fixed point theorem, but domain theorists ought to have found it much earlier.

What we did learn from the intuitionistic ordinals [JM95, Tay96a] is that their irreflexive membership and reflexive containment relations must be considered separately. In symbolic logic this at least doubles the work, but category theory was invented to *organise* such difficulties, by isolating the essential *argument*, whilst wrapping the *complications* in an appropriate choice of categories and functors.

Therefore the next task is to apply the present work to the category of posets instead of sets [Tay25c]; we haven't included that here because there are too many order-theoretic facts and fallacies to check. Then the ideas can be extended to other categories, which might have fixpoint objects [CP92] or accommodate corecursion alongside recursion, *cf.* Remark 4.6.

This is why we went to some trouble in Section 4 to pin down just what we were using in the original setting. In other categories there are many alternatives to the naïve ideas of 1–1 and onto mappings that we could use for the predicates over which we do induction and for defining extensionality .

Extensionality is not as innocent as it looks: equality is like marriage in that it transfers any property of one partner to the other. Dana Scott showed that it is essential for giving the axiom of replacement its power: *without* it, that is provably consistent in *Zermelo* Set Theory [Sco66].

We expect to see even more dramatic results from applying the extensional reflection (Theorem 8.13) to other categorical settings. Using various notions of "initial segment" and "cofinal map" turns sets ($\in$-structures) into ordinals and thin ordinals into plump ones. Transfinite iteration of functors is also an example of this process.

To perform these over Set Theory requires Replacement, but, being adjoints, they are expressed in the mother tongue of category theory, so we can regard them as *candidates for new axioms* to replace Replacement.

We have also explained how extensional well founded coalgebras are "fragments" of the initial algebra, whether that exists or not. Even if it does, it may be very complicated, whilst it may be easier to characterise its fragments instead.

Since there is plenty to do in "concrete" categories, it is not really an issue that we haven't fully explained how they are well powered. Any fibration defines a factorisation system, in which we would require prone maps to be initial segments and cofinal ones to be vertical. If we are going that deep into foundations, we should also deconstruct what is needed of the base category $\mathcal{S}$ to prove Pataraia's Theorem, in particular the directed completeness and impredicativity.

All of these considerations come together when the algebra is some type theory. There is a categorical construction called *gluing* [Tay99, §7.7] or *logical relations* that apparently magically proves consistency and termination results. It invokes the universal property of the free algebra, *i.e. recursion* over the *entirety* of its world of types, terms and proofs. How it manages to do this ought to raise eyebrows in the light of Kurt Gödel's incompleteness theorems.

The symbolic approach to such things is to turn the syntax of proofs into an ordinal, which to a categorist is vandalism because it throws the algebraic structure away. In fact proof theorists also exploit their *arithmetic* of ordinals to keep track of iterated transformations of proofs. One might hope to develop methods that retain both the algebra of the type theory and that of proof-manipulation.

Above all we must escape from the idea that ordinals are linear orders for counting beyond infinity.

The earliest version of this work was presented at *Category Theory 1995* in Cambridge and at *Logical Foundations of Mathematics, Computer Science and Physics — Kurt Gödel's Legacy (Gödel '96)* in Brno. Although it did not appear in the proceedings of either meeting, [Tay96b] was circulated there and available on my web page from 1996 to 2003 and from 2006. Summaries of the results were published in Sections 2.5, 6.3, 6.7 and 9.5 of [Tay99]. Work was resumed in 2019 in answer to a demand from those studying coalgebras to weaken the conditions on the functor.

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