The Limit-Colimit Coincidence for Categories

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Abstract

Scott noticed in 1969 that in the category of complete lattices and maps preserving directed joins, limit of a sequence of projections (maps with preinverse left adjoints) is isomorphic to the colimit of those left adjoints (called embeddings). This result holds in any category of domains and is the basis of the solution of recursive domain equations. Limits of this kind (and continuity with respect to them) also occur in domain semantics of polymorphism, where we find that we need to generalise from sequences to directed or filtered diagrams and drop the "preinverse" condition. The result is also applicable to the proof of cartesian closure for stable domains.

The purpose of this paper is to express and prove the most general form of the result, which is for filtered diagrams of adjoint pairs between categories with filtered colimits. The ideas involved in the applications are to be found elsewhere in the literature: here we are concerned solely with 2-categorical details. The result we obtain is what is expected, the only remarkable point being that it seems definitely to be about *pseudo-* and not *lax* limits and colimits.

On the way to proving the main result, we find ourselves also performing the constructions needed for domain interpretations of dependent type polymorphism.

1 Filtered diagrams

The result concerns limits and colimits of filtered diagrams of categories, so we shall be interested in functors of the form

$$\mathcal{I} \to \mathbf{FCCat}^{\mathsf{cp}}$$

where $\mathcal{I}$ is a filtered category and $\mathbf{FCCat}^{\mathsf{cp}}$ is the *2-category* of small categories with filtered colimits, functors with right adjoints which preserve filtered colimits and natural transformations. Since we have to perform a lot of manipulation of filtered diagrams, we begin with a short discussion on them.

Definition 1.1 A category (diagram-type) $\mathcal{I}$ is *filtered* if

(i) it is non-empty,

(ii) it has the amalgamation property for objects, *i.e.* given $i, j \in \mathcal{I}$, there is some $k \in \mathcal{I}$ and $u : i \to k$, $v : j \to k$ in $\mathcal{I}$.

(iii) it has the amalgamation property for morphisms, *i.e.* given a parallel pair $u, v : i \rightrightarrows j$ in $\mathcal{I}$, there is some $k \in \mathcal{I}$ and $w : j \to k$ in $\mathcal{I}$ with $u \,;\, w = v \,;\, w$.

Example 1.2 Any category with finite colimits is filtered, for example ordinals and finite power-set lattices. The definition says that a filtered category has "weak" finite colimits.

Part of the data for the theorem is that the categories $\mathcal{X}^i$ "have filtered colimits." We shall always take this to mean that there is a *functor*

$$\operatorname*{\mathsf{colim}^{\uparrow}}_{\mathcal{J}} : \mathcal{X}^{\mathcal{J}} \to \mathcal{X}$$

which is left adjoint to the "constant functor"; the counit is the *colimiting cocone* and the unit (since $\mathcal{J}$ is connected) is an isomorphism. Often, however, we can avoid naming the colimit because we are given a cocone and set out to prove that it is colimiting, *i.e.* *a* colimit rather than *the chosen* colimit.

Definition 1.3 A functor $F : \mathcal{X} \to \mathcal{Y}$ is *continuous* if it preserves (commutes with) filtered colimits *up to isomorphism* ϕ:

$$\begin{tikzcd} \operatorname{colim}^{\uparrow}_{j\in\mathcal{J}} FX_j \arrow[r, "\phi_{\mathcal{J}}\langle X_j\rangle", "\cong"'] & F\Big(\operatorname{colim}^{\uparrow}_{j\in\mathcal{J}} X_j\Big) \\ FX_j \arrow[u, "\sigma_j"] \arrow[r, equal] & FX_j \arrow[u, "F\tau_j"'] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{X}^{\mathcal{J}} \arrow[r, "{}^{\mathcal{X}}\operatorname{colim}^{\uparrow}_{\mathcal{J}}"] \arrow[d, "F^{\mathcal{J}}"'] \arrow[dr, phantom, "\overset{\phi}{\Rightarrow}"] & \mathcal{X} \arrow[d, "F"] \\ \mathcal{Y}^{\mathcal{J}} \arrow[r, "{}^{\mathcal{Y}}\operatorname{colim}^{\uparrow}_{\mathcal{J}}"'] & \mathcal{Y} \end{tikzcd}$$

We shall make several uses of *final subdiagrams*, which generalise the idea of a cofinal subsequence of an ordinal. They is dealt with in [Mac Lane], section 9.3.

Definition 1.4 A functor $U : \mathcal{J} \to \mathcal{I}$ is *final* if for each $i \in \mathcal{I}$ the comma category $i \downarrow U$ is connected. This means that there is some $j \in \mathcal{J}$ and $u : i \to Uj$, and if there is another $j' \in \mathcal{J}$ with $u' : i \to Uj'$ then there is a "zig-zag" such that

$$\begin{tikzcd} i \arrow[r, equal] \arrow[d, "u=u_0"'] & i \arrow[r, equal] \arrow[d, "u_1"'] & i \arrow[r, equal] \arrow[d, "u_2"'] & i \arrow[r, equal] \arrow[d, "u_{2n-1}"] & i \arrow[d, "u_{2n}=u'"] \\ Uj = Uj_0 \arrow[r, "Uv_1"] & Uj_1 & Uj_2 \arrow[l, "Uv_2"'] \;\cdots & Uj_{2n-1} & Uj_{2n} = Uj' \arrow[l, "Uv_{2n}"'] \end{tikzcd}$$

commutes.

Proposition 1.5 Let $D : \mathcal{I} \to \mathcal{X}$ be a diagram in a category and $U : \mathcal{J} \to \mathcal{I}$ be a final functor. Suppose that the diagram $DU : \mathcal{J} \to \mathcal{I} \to \mathcal{X}$ has a colimit with cocone $y_j : DUj \to X$. Then there is a unique cocone $x_i : Di \to X$ over the given diagram D such that $x_{Uj} = y_j$, and this is colimiting.

Proof For $i \in \mathcal{I}$, suppose $j \in \mathcal{J}$ with $u : i \to Uj$, and put $x_i = Du\,; y_j$. We have to check that this is well-defined, by induction on the length of a zig-zag between two candidates. It suffices to consider the case of a single "zig", $u' = u;Uv$, where $v : j \to j'$; then $x_i = Du;y_j = Du;DUv;y_{j'} = Du'\,; y_{j'}$. This also shows that x_i is a cocone over $\mathcal{I}$. It is clearly also the only one with $x_{Uj} = y_j$.

Now let $z_i : Di \to Z$ be another cocone over $\mathcal{I}$. Then $z_{Uj} : DUj \to Z$ is a cocone over $\mathcal{J}$ and, since X is the colimit, there is a unique mediator $z : X \to Z$ with $y_j\,; z = z_{Uj}$. Then $x_i\,; z = Du\,; y_j\,; z = Du\,; z_{Uj} = z_i$, so z mediates from $\mathcal{I}$ also, and *a fortiori* it is unique. $\square$

Definition 1.6 We shall need the following constructions on a fixed filtered category $\mathcal{I}$ (with typical objects):

(i) $\mathcal{I}$ itself

$$i$$

(ii) the *coslice* $i_0/\mathcal{I}$

$$i_0 \xrightarrow{u} i$$

(iii) the *double coslice* $(i_0, j_0)/\mathcal{I}$

(iv) the *degenerate double coslice* $(i_0, i_0)/\mathcal{I}$

$$\begin{tikzcd} i_0 \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & i \end{tikzcd}$$

(v) [a construction needed in Lemma 8.2]...

Exercise 1.7 Let $\mathcal{I}$ be a filtered category. Verify that the following functors are final:

(a) $i_0/\mathcal{I} \to (i_0, i_0)/\mathcal{I}$ by $u \mapsto (u, u)$;

(b) $(i_0, j_0)/\mathcal{I} \to$ (v);

(c) $i_0/\mathcal{I} \to i_1/\mathcal{I}$ by $u \mapsto p\,;u$;

(d) $(i_0, j_0)/\mathcal{I} \to (i_1, j_1)/\mathcal{I}$ by $(u, v) \mapsto (p\,;u, q\,;v)$;

(e) $\mathcal{I} \to \mathcal{I} \times \mathcal{I}$ by $i \mapsto (i, i)$;

where $i_0, i_1, j_0, j_1 \in \mathcal{I}$ and $p : i_1 \to i_0$ and $q : j_1 \to j_0$ in $\mathcal{I}$. $\square$

Exercise 1.8 Suppose $\mathcal{I}$ has binary coproducts. Show that the functors

(a) $\mathcal{I} \to i_0/\mathcal{I}$ by $i \mapsto (i_0 \to i_0 + i)$, and

(a) $\mathcal{I} \to (i_0, j_0)/\mathcal{I}$ by $i \mapsto (i_0 \to i_0 + j_0 + i \leftarrow j_0)$

(the maps being the coproduct inclusions) are final. Hence show that any colimit over $i_0/\mathcal{I}$, $(i_0, j_0)/\mathcal{I}$ or $\mathcal{I} \times \mathcal{I}$ may be calculated as a colimit over $\mathcal{I}$. $\square$

So the above assumptions about "categories with filtered colimits" may be reduced to "categories with colimits of type $\mathcal{I}$". Indeed, if we interpret the definition of filteredness constructively, any filtered category is equivalent (for the purpose of finding colimits) to a category with finite colimits.

Although we said that we wanted the categories to have *all* filtered colimits, it turns out that we only actually make use of colimits of diagrams of the above types. Clearly these are of essentially the same cardinality as the given diagram $\mathcal{I}$, which allows us to regard filtered colimits as an algebraic operation of fixed arity. We also never consider the aggregate of all categories with filtered colimits, or perform constructions worse than products of the cardinality of $\mathcal{I}$. Consequently we have no problems of size, and may treat the categories as small (or locally small); this is fortunate, considering the amount of 2-categorical work already cut out for us!

2 Pseudofunctors

In this section we shall examine the notion of pseudofunctor $\mathcal{X}^{(-)} : \mathcal{I}^{\mathrm{op}} \to \mathbf{Cat}$. Although in the application to the limit-colimit coincidence (sections 5 and 7 onwards) we shall need $\mathcal{I}$ to be filtered, for the time being it may be any category whatever.

The problem is that in general for composable arrows $u : i \to j$ and $v : j \to k$ the functors

$$\mathcal{X}^{u;v} \quad \text{and} \quad \mathcal{X}^u\mathcal{X}^v$$

cannot be expected to be equal: at best isomorphic. This is because $\mathcal{X}^u$ is typically defined by some universal property, such as pullback along u. [More generally, one may study the case where they are related by a *non-invertible* map one way or the other, and such constructions are called *lax* or *oplax*.] We shall find, however, that in the result which interests us we need an isomorphism

$$\mathcal{X}^u\mathcal{X}^v \xrightarrow{\alpha^{u,v}} \mathcal{X}^{u;v}$$

Likewise, there is no *a priori* reason why $\mathcal{X}^{\mathsf{id}}$ need be the identity, so we also provide an isomorphism

$$\mathsf{id}_{\mathcal{X}^i} \xrightarrow{\gamma^i} \mathcal{X}^{\mathsf{id}_i}$$

(We prefer for reasons of symmetry not to put $\gamma^i = \mathsf{id}$, although this is possible and quite commonly done.) Since these maps are isomorphisms, we shall feel free to define compositions involving them backwards or forwards, without writing α^{-1}.

Definition 2.1 A *pseudofunctor* $\mathcal{I}^{\mathrm{op}} \to \mathbf{Cat}$ is an assignment of categories $\mathcal{X}^i$, functors $\mathcal{X}^u : \mathcal{X}^j \to \mathcal{X}^i$ and natural isomorphisms $\alpha^{u,v}$ and γ^i as above, subject to the laws

$$\begin{tikzcd}
\mathcal{X}^u\mathcal{X}^v\mathcal{X}^w \arrow[r, "\mathcal{X}^u\alpha^{v,w}"] \arrow[d, "\alpha^{u,v}\mathcal{X}^w"'] \arrow[dr, phantom, "\mathsf{A}"] & \mathcal{X}^u\mathcal{X}^{v;w} \arrow[d, "\alpha^{u,v;w}"] \\
\mathcal{X}^{u;v}\mathcal{X}^w \arrow[r, "\alpha^{u;v,w}"] & \mathcal{X}^{u;v;w}
\end{tikzcd} \qquad \text{associativity}$$

and

$$\begin{tikzcd}
\mathcal{X}^u \arrow[r, "\gamma^i\mathcal{X}^u"] \arrow[dr, "\mathsf{id}_{\mathcal{X}^u}"'] \arrow[dr, phantom, "\mathsf{L}", shift left=2] & \mathcal{X}^{\mathsf{id}_i}\mathcal{X}^u \arrow[d, "\alpha^{\mathsf{id}_i,u}"] \\
& \mathcal{X}^u
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{X}^u\mathcal{X}^{\mathsf{id}_j} \arrow[d, "\alpha^{u,\mathsf{id}_j}"'] & \mathcal{X}^u \arrow[l, "\mathcal{X}^u\gamma^j"'] \arrow[dl, "\mathsf{id}_{\mathcal{X}^u}"] \arrow[dl, phantom, "\mathsf{R}", shift right=2] \\
\mathcal{X}^u &
\end{tikzcd} \qquad \text{unit}$$

Lemma 2.2 Any well-formed diagram consisting of α's, γ's and their inverses commutes.

Proof By induction on the length of the compositions, using the associativity and unit laws. □

Corollary 2.3 If $\vec{u}$ and $\vec{v}$ are composable strings in $\mathcal{I}$ with a common refinement, then there is a *unique* canonical isomorphism between $\mathcal{X}^{\vec{u}} \stackrel{\mathrm{def}}{=} \mathcal{X}^{u_1}...\mathcal{X}^{u_n}$ and $\mathcal{X}^{\vec{v}}$. □

This kind of lengthy presentation of data conflicts with the spirit of category theory, and we have two alternatives: 2-functors and fibrations. (*What about sheaves of categories?*)

Construction 2.4 A 2-category $\mathbf{I}$ whose 0-cells are those of $\mathcal{I}$ and whose 1-cells are composable strings of arrows of $\mathcal{I}$. There is a unique, invertible, 2-cell between any two strings a common refinement. We then obtain a 2-functor (no longer pseudo)

$$d : \mathbf{I}^{\mathrm{op}} \to \mathbf{Cat} \qquad \text{by } (u_1, ..., u_n) \mapsto \mathcal{X}^{u_1}...\mathcal{X}^{u_n}$$

whose effect on 2-cells is well-defined by the corollary. □

Proposition 2.5 There is an isomorphism between pseudofunctors $\mathcal{I}^{\mathrm{op}} \to \mathbf{Cat}$ and 2-functors $\mathbf{I}^{\mathrm{op}} \to \mathbf{Cat}$. □

Notation 2.6 Besides the equations A, L, R, *etc.*, we shall denote the naturality square of $\phi : F \to G$ with respect to $f : X \to Y$ by

$$\begin{tikzcd} FX \arrow[r, "\phi X"] \arrow[d, "Ff"'] \arrow[dr, phantom, "\mathsf{N}"] & GX \arrow[d, "Gf"] \\ FY \arrow[r, "\phi Y"] & GY \end{tikzcd} \quad \text{or} \quad \begin{tikzcd} FX \arrow[r, "Ff"] \arrow[d, "\phi X"'] \arrow[dr, phantom, "\mathsf{Z}"] & FY \arrow[d, "\phi Y"] \\ GX \arrow[r, "Gf"] & GY \end{tikzcd}$$

3 Fibrations

The other approach, sometimes called the *Grothendieck construction*, is to form the total category $\Sigma i : \mathcal{I}.X^i$ and display it as a fibration over $\mathcal{I}$.

Construction 3.1 The *total category* $\Sigma i : \mathcal{I}.\mathcal{X}^i$.

(i) The objects of the total category are the pairs (i, X) with $i \in \mathcal{I}$ and $X \in \mathcal{X}^i$;

(ii) the morphisms from (i, X) to (j, Y) are the pairs (u, f) where $u : i \to j$ in $\mathcal{I}$ and $f : X \to \mathcal{X}^u Y$;

(iii) the identity on (i, X) is $(\mathsf{id}_i, \gamma^i X)$;

(iv) the composite of $(u, f) : (i, X) \to (j, Y)$ and $(v, g) : (j, Y) \to (k, Z)$ is $(u \,;v,\ f \,; \mathcal{X}^u g \,; \alpha^{u,v})$.

The *display functor* $P : \Sigma i : \mathcal{I}.\mathcal{X}^i \to \mathcal{I}$

(v) takes (i, X) to i and (u, f) to u.

The *inclusion functor* $I_{i_0} : \mathcal{X}^{i_0} \to \Sigma i : \mathcal{I}.\mathcal{X}^i$

(vi) takes X to (i_0, X) and $f : X \to Y$ to $(\mathsf{id}_{i_0},\ f \,; \gamma^{i_0} Y)$.

The *coherence* $\upsilon_u : I_i \mathcal{X}^u \to I_j$

(vii) at $Y \in \mathcal{X}^j$ is $(u, \mathsf{id}\mathcal{X}^u Y)$ and satisfies the equations

$$\begin{tikzcd} I_i \arrow[r, "I_i \gamma^i"] \arrow[dr, "\mathsf{id}"'] \arrow[dr, phantom, "\mathsf{U}", shift left=2] & I_i \mathcal{X}^{\mathsf{id}} \arrow[d, "\upsilon_{\mathsf{id}}"] \\ & I_i \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} I_i \mathcal{X}^u \mathcal{X}^v \arrow[r, "\upsilon_u \mathcal{X}^v"] \arrow[d, "I_i \alpha^{u,v}"'] \arrow[dr, phantom, "\mathsf{M}"] & I_j \mathcal{X}^v \arrow[d, "\upsilon_v"] \\ I_i \mathcal{X}^{u;v} \arrow[r, "\upsilon_{u;v}"] & I_k \end{tikzcd}$$

Proof First, $\Sigma i : \mathcal{I}.\mathcal{X}^i$ is a category. For the identities:

$$\begin{aligned} (u, f) \,; \mathsf{id}_{(j,Y)} &= (u \,; \mathsf{id}_j,\ f \,; \mathcal{X}^u \gamma^j Y \,; \alpha^{u,\mathsf{id}} Y) \\ &= (u \,; \mathsf{id}_j,\ f \,; \mathsf{id}_Y) \\ &= (u, f) \end{aligned}$$

using R and

$$\begin{aligned} \mathsf{id}_{(i,X)} \,; (u, f) &= (\mathsf{id}_i \,; u,\ \gamma^i X \,; \mathcal{X}^{\mathsf{id}} f \,; \alpha^{\mathsf{id},u} Y) \\ &= (\mathsf{id}_i \,; u,\ f \,; \gamma^i \mathcal{X}^u Y \,; \alpha^{\mathsf{id},u} Y) \\ &= (u, g \,; \mathsf{id}) \end{aligned}$$

using naturality and L. Associativity follows from:

$$\begin{array}{rcll} \mathcal{X}^u(\mathcal{X}^v h\,;\alpha^{v,w}T) &=& \mathcal{X}^u\mathcal{X}^v h\,;\alpha^{u,v}\mathcal{X}^w T\,;\alpha^{u;v,w}T & \mathsf{A}\\ &=& \alpha^{u,v}Z\,;\mathcal{X}^{u;v}h\,;\alpha^{u;v,w}T & \mathsf{N}\end{array}$$

Functoriality of P is obvious, as is the fact that I_i preserves identities; for composition:

$$\begin{array}{rcll} I_i f\,;I_i g &=& (\mathsf{id}_i, f\,;\gamma^i Y)\,;(\mathsf{id}_i, g\,;\gamma^i Z) & \\ &=& \big(\mathsf{id}_i, f\,;(\gamma^i Y\,;\mathcal{X}^{\mathsf{id}}g)\,;(\mathcal{X}^{\mathsf{id}}\gamma^i Z\,;\alpha^{\mathsf{id},\mathsf{id}}Z)\big) & \\ &=& (\mathsf{id}_i, f\,;g\,;\gamma^i Z\,;\mathsf{id}_{\mathcal{X}^{\mathsf{id}}Z}) & \mathsf{Z}, \mathsf{R}\end{array}$$

Naturality of υ:

$$\begin{tikzcd}
\mathcal{X}^u Y \arrow[r, "\mathsf{id}"] \arrow[d, "\mathcal{X}^u y"'] & \mathcal{X}^u Y \arrow[d, "\mathcal{X}^u y"] & \\
\mathcal{X}^u Y' \arrow[d, "\gamma^i\mathcal{X}^u Y'"'] \arrow[dr, "\mathsf{id}"] \arrow[dr, phantom, "\mathsf{L}"', near start] & \mathcal{X}^u Y' \arrow[r, "\mathcal{X}^u\gamma^j Y'"] \arrow[d, "\mathsf{id}"'] & \mathcal{X}^u\mathcal{X}^{\mathsf{id}}Y' \arrow[dl, "\alpha^{u,\mathsf{id}}Y'"] \arrow[dl, phantom, "\mathsf{R}"', near start] \\
\mathcal{X}^{\mathsf{id}}\mathcal{X}^u Y' \arrow[r, "\alpha^{\mathsf{id},u}Y"'] & \mathcal{X}^u Y' &
\end{tikzcd}$$

The U equation is essentially just L, and for M:

$$\begin{tikzcd}
\mathcal{X}^u\mathcal{X}^v \arrow[rr, "\mathsf{id}"] \arrow[d, "\alpha^{u,v}"'] \arrow[dr, "\gamma^i\mathcal{X}^u\mathcal{X}^v"] & \arrow[d, phantom, "\mathsf{L}"] & \mathcal{X}^u\mathcal{X}^v \arrow[d, "\mathsf{id}"] \\
\mathcal{X}^{u;v} \arrow[r, phantom, "\mathsf{N}"] \arrow[d, "\gamma^i\mathcal{X}^{u;v}"'] & \mathcal{X}^{\mathsf{id}}\mathcal{X}^u\mathcal{X}^v \arrow[r, "\alpha^{\mathsf{id},u}\mathcal{X}^v"] \arrow[d, "\mathcal{X}^{\mathsf{id}}\alpha^{u,v}"'] \arrow[dr, phantom, "\mathsf{A}"] & \mathcal{X}^u\mathcal{X}^v \arrow[d, "\alpha^{u,v}"] \\
\mathcal{X}^{\mathsf{id}}\mathcal{X}^{u;v} \arrow[r, "\mathsf{id}"] & \mathcal{X}^{\mathsf{id}}\mathcal{X}^{u;v} \arrow[r, "\alpha^{\mathsf{id},u;v}"] & \mathcal{X}^{u;v}
\end{tikzcd}$$

□

Definition 3.2 Let $P : \mathcal{S} \to \mathcal{I}$ be a functor. A morphism $g : Y \to Z$ in $\mathcal{S}$ is

(i) *vertical* if $PY = PZ$ and $Pg = \mathsf{id}_{PY}$, and

(ii) *horizontal* over (*i.e.* mapping to) $v : j \to k$ (where $j = PY$, *etc.*) if for every $h : X \to Z$ over $u\,;v$ there is a unique $f : X \to Y$ over u with $h = f\,;g$.

The functor P is called a *fibration* if for every $v : j \to k$ and Z with $PZ = k$ there is some horizontal $g : Y \to Z$ over v.

Lemma 3.3 $P : \Sigma i : \mathcal{I}.X^i \to \mathcal{I}$ is a fibration, in which (v, g) is vertical iff $v = \mathsf{id}$ and horizontal iff g is invertible, the horizontal lifting of v at Z being $\upsilon_v Z$.

Proof The only part which isn't obvious is that if (v, g) is horizontal then g is invertible (for the converse recall that $\alpha^{u,v}$ is also invertible); for this we use horizontality twice with $u = \mathsf{id}_i$. First let $X_1 = \mathcal{X}^v Z$, $h_1\mathsf{id}$ and derive f_1 such that $[f_1 \,;(\gamma^j Y)^{-1}] \,; g = \mathsf{id}$. Then put $X_2 = Y$, $h_2 g$; now id and $g \,; [f_1 \,; (\gamma^j Y)^{-1}]$ both serve for f_2. □

Lemma 3.4 If $P : \mathcal{S} \to \mathcal{I}$ is a fibration then there is a factorisation system in $\mathcal{S}$ consisting of the (*pseudo!)vertical and horizontal maps.

Proof Let $h : X \to Z$ over $u : i \to j$. Let $g : Y \to Z$ be a horizontal lifting of u at Z and $f : X \to Y$ the unique vertical map with $h = f \,; g$. Now if we are given $v \,; h = k \,; g$ with v vertical and g horizontal, then h is over $Ph = Pk \,; Pg$ and so there is a unique f over Pk with $f \,; g = h$; likewise $v \,; h$ is over $Ph = Pk \,; Pg$ and $v \,; f$ and k both serve as the mediator, so $v \,; f = k$. □

Exercise 3.5 Conversely, if P is any functor satisfying the definition of fibration with the word "horizontal" deleted, and the vertical and horizontal maps in $\mathcal{S}$ form a factorisation system, then P is a fibration.

Proposition 3.6 $\Sigma i : \mathcal{I}.\mathcal{X}^i$ is the *lax colimit* of the diagram, *i.e.* if $N_j : \mathcal{X}^j \to \mathcal{Y}$ are functors and $n_u : N_i \mathcal{X}^u \to N_j$ natural transformations such that

$$\mathsf{id} = N_i \gamma^i \,; n_{\mathsf{id}} \quad \text{and} \quad n_u \mathcal{X}^v \,; n_v = N_i \alpha^{u,v} \,; n_{u;v}$$

then there is a unique functor $N : \Sigma i : \mathcal{I}.\mathcal{X}^i \to \mathcal{Y}$ such that

$$NI_i = N_i \quad \text{and} \quad N\upsilon_u = n_u$$

Proof On objects, we are forced to put $N(i, X) = N_i X$. For morphisms we use the vertical-horizontal factorisation: $(u, f) = I_i f \,; \upsilon_u Y$, so we must put $N(u, f) = N_i f \,; n_u Y$. Universality of the factorisation makes this *functorial. □

Now for the converse; clearly the category $\mathcal{X}^i$ has objects those $X \in \mathcal{S}$ with $PX = i$ and morphisms the vertical maps over i. However, since horizontal maps are defined by a universal property, the lifting of $v : j \to k$ at Z with $PZ = k$ is defined only up to isomorphism. We therefore have to make a *choice* of liftings in order to determine a functor $\mathcal{X}^v : \mathcal{X}^k \to \mathcal{X}^j$.

Lemma 3.7 $\mathcal{X}^v$ is *pseudofunctorial in v. □

Proposition 3.8 There is a weak equivalence from pseudofunctors $\mathcal{I}^{\mathrm{op}} \to \mathbf{Cat}$ to fibrations $- \to \mathcal{I}$. □

4 Oplax and pseudo limits

Having established what we mean by a (pseudo)diagram of categories, we can now calculate its pseudolimit. Although we have chosen not to consider lax *diagrams*, we can still form a lax or oplax *limit* of the diagram, which corresponds to a Π-type. This has application to the construction of filtered colimits in Σ-types.

Construction 4.1 Let $\mathcal{X}^{(-)} : \mathcal{I}^{\mathrm{op}} \to \mathbf{Cat}$ be a pseudofunctor. The categories $\mathsf{Lim}\, i : \mathcal{I}.\mathcal{X}^i$ and $\Pi i : \mathcal{I}.\mathcal{X}^i$ have

(i) *objects* the families $\langle X^i, x^u \rangle$ with $X^i \in \mathcal{X}^i$ for $i \in \mathcal{I}$ and $x^u : X^i \to \mathcal{X}^u X^j$ for $u : i \to j$ such that

$$\begin{tikzcd}
X^i \arrow[r, "\gamma^i X^i"] \arrow[dr, "\mathsf{id}"'] \arrow[dr, phantom, shift left=3, "\mathsf{U}"] & \mathcal{X}^{\mathsf{id}_i} X^i \\
& X^i \arrow[u, "x^{\mathsf{id}_i}"']
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathcal{X}^u \mathcal{X}^v X^k \arrow[r, "\alpha^{u,v} X^k"] \arrow[dr, phantom, "\mathsf{M}"] & \mathcal{X}^{u;v} X^k \\
\mathcal{X}^u X^j \arrow[u, "\mathcal{X}^u x^v"] & X^i \arrow[l, "x^u"'] \arrow[u, "x^{u;v}"']
\end{tikzcd}$$

commute, where x^u is respectively an isomorphism and arbitrary;

(ii) *morphisms* from $\langle X^i, x^u\rangle$ to $\langle Y^i, y^u\rangle$ the families $\langle f^i : X^i \to Y^i\rangle$ for $i \in \mathcal{I}$ such that

$$\begin{tikzcd} \mathcal{X}^u X^j \arrow[r, "\mathcal{X}^u f^j"] & \mathcal{X}^u Y^j \\ X^i \arrow[u, "x^u"] \arrow[r, "f^i"'] \arrow[ur, phantom, "\mathsf{H}"] & Y^i \arrow[u, "y^u"'] \end{tikzcd}$$

commutes;

(iii) componentwise identities and composition.

(iv) There are *projection functors* to $\mathcal{X}^i$ given by the extraction of i-components, for which we also write $\mathcal{X}^i$, and

(v) *coherences* $\xi^u : \mathcal{X}^i \to \mathcal{X}^u\mathcal{X}^j$ given by extraction of u-components, which satisfy the equations U and M

$$\gamma^i\mathcal{X}^i = \xi^{\mathsf{id}_i} \quad \text{and} \quad \xi^{u;v} = \xi^u \,;\, \mathcal{X}^u\xi^v \,;\, \alpha^{u,v}\mathcal{X}^k$$

analogous to part (i) above.

Proof Obvious. □

Lemma 4.2 $\Pi i : \mathcal{I}.\mathcal{X}^i$ is isomorphic to the category of *sections* of the display functor $\Sigma i : \mathcal{I}.\mathcal{X}^i \to \mathcal{I}$.

Proof A section is a functor $S : \mathcal{I} \to \Sigma i : \mathcal{I}.\mathcal{X}^i \to \mathcal{I}$ such that $PS = \mathsf{id}$. This means that

$$Si = (i, X^i) \quad \text{and} \quad Su = (u, x^u)$$

where $X^i \in \mathcal{X}^i$ and $x^u : X^i \to \mathcal{X}^u X^j$. Functoriality corresponds exactly to the equations U and M. Likewise, a natural transformation $\phi : S \to T$ between sections satisfies $P\phi = \mathsf{id}$, so

$$\phi i = (\mathsf{id}_i, f^i)$$

where naturality makes $f^i : X^i \to Y^i$ satisfy the equation H. □

Proposition 4.3 $\Pi i : \mathcal{I}.\mathcal{X}^i$ is the *oplax limit* and $\mathsf{Lim}\, i : \mathcal{I}.\mathcal{X}^i$ is the *pseudolimit.*

Proof Given functors $F^i : \mathcal{A} \to \mathcal{X}^i$ and natural transformations $\sigma^u : F^i \to \mathcal{X}^u F^j$ such that the equations U and M hold:

$$\gamma^i F^i = \sigma^{\mathsf{id}_i} \quad \text{and} \quad \sigma^{u;v} = \sigma^u \,;\, \mathcal{X}^u\sigma^v \,;\, \alpha^{u,v}F^k$$

then there is a unique functor $F : \mathcal{A} \to \Pi i : \mathcal{I}.\mathcal{X}^i$ such that

$$\mathcal{X}^i F = F^i \quad \text{and} \quad \xi^u F = \sigma^u$$

On objects we are forced to put $FA = \langle F^i A, \sigma^u A\rangle$, and this is an object of the oplax limit by equations U and M. Likewise on morphisms we have $Fa = \langle F^i a\rangle$, with H given by N. Then F is obviously a functor and unique. Moreover σ are isomorphisms, then F factors through the inclusion $\mathsf{Lim}\, i : \mathcal{I}.\mathcal{X}^i \subset \Pi i : \mathcal{I}.\mathcal{X}^i$. □

Proposition 4.4 The limits are **Cat**-*enriched*: if $\phi^i : F^i \to G^i$ is a compatible family of natural transformations, *i.e.* the square

$$\begin{tikzcd}
F^i \arrow[r, "\phi^i"] \arrow[d, "\sigma^u"'] \arrow[dr, phantom, "\mathsf{H}"] & G^i \arrow[d, "\tau^u"] \\
\mathcal{X}^u F^j \arrow[r, "\mathcal{X}^u \phi^j"] & \mathcal{X}^u G^j
\end{tikzcd}$$

commutes, then there is a unique natural transformation $\phi : F \to G$ such that $\phi^i = \mathcal{X}^i \phi$.

Proof Obvious. $\square$

5 Continuous fibrations

In this section we assume that each $\mathcal{X}^j$ has (filtered) colimits of types $\mathcal{I}$ and $j/\mathcal{I}$ and that these are preserved by $\mathcal{X}^u$.

Proposition 5.1 The pseudo and oplax limits have filtered colimits, which are computed componentwise and preserved by the projection functors $\mathcal{X}^i$.

Proof Let $\mathcal{J}$ be a filtered diagram for which each $\mathcal{X}^i$ has and $\mathcal{X}^u$ preserves colimits. For $j \in \mathcal{J}$ let $\langle X^i_j, x^u_j \rangle$ be an object of the oplax or pseudo limit, and for $v : j \to j'$ let $\langle f^i_v : X^i_j \to X^i_{j'} \rangle$ be a morphism, so that we have a diagram of type $\mathcal{I}$ in the limit. Then we also have diagrams in each $\mathcal{X}^i$, and we may let $f^i_j : X^i_j \to X^i$ be a colimiting cocone.

Define x^u as the unique map making

$$\begin{tikzcd}
X^i \arrow[r, "x^u"] & \mathcal{X}^u X^{i'} \\
X^i_j \arrow[r, "x^u_j"] \arrow[u, "f^i_j"] & \mathcal{X}^u X^{i'}_j \arrow[u, "\mathcal{X}^u f^{i'}_j"']
\end{tikzcd}$$

commute for each $j \in \mathcal{J}$. If each x^u_j is invertible then so is x^u, so the result specialises from the oplax to the pseudo limit. Once we have shown that $\langle X^i, x^u \rangle$ is an object of the limit, this diagram immediately makes f_j a morphism, and it belongs to a cocone because each f^i_j does.

The *equations U and M for the object $\langle X^i, x^u \rangle$ are simply the colimits of the same equations for $\langle X^i_j, x^u_j \rangle$.

If $\langle g^i_j : X^i_j \to Z^i \rangle$ is another cocone, then by the componentwise colimits we have a unique mediator $g^i : X^i \to Z$, and this is a *morphism $g = \langle g^i \rangle$ by uniqueness of mediators $X^i \to \mathcal{X}^u Z^{i'}$. $\square$

Construction 5.2 Let $\langle X^i, x^u \rangle \in \Pi i : \mathcal{I}.\mathcal{X}^i$. A diagram of type $i/\mathcal{I}$ in $\mathcal{X}^i$ which takes

$$(i \xrightarrow{u} j) \xrightarrow{v} (i \xrightarrow{u;v} k) \qquad \text{to} \qquad \mathcal{X}^u X^j \xrightarrow{\mathcal{X}^u x^v} \mathcal{X}^u \mathcal{X}^v X^k \xrightarrow{\alpha^{u,v} X^k} \mathcal{X}^{u;v} X^k$$

Proof We have to check functoriality:

$$\begin{tikzcd}
\mathcal{X}^u X^i \arrow[r, "\mathcal{X}^u x^{\mathsf{id}}"] & \mathcal{X}^u \mathcal{X}^{\mathsf{id}} X^i \arrow[d, "\alpha^{u,\mathsf{id}} X^k"] \\
\mathcal{X}^u X^i \arrow[u, "\mathsf{id}"] \arrow[ur, "\gamma^i" description] \arrow[r, "\mathsf{id}"] & \mathcal{X}^u X^i
\end{tikzcd}$$

with regions labelled U (upper left) and R (lower right).

and

$$
\begin{tikzcd}
\mathcal{X}^u X^j \arrow[r, "\mathcal{X}^u x^v"] \arrow[dd, "\mathcal{X}^u x^{v;w}"] \arrow[dr, phantom, "\mathsf{M}"] & \mathcal{X}^u \mathcal{X}^v X^k \arrow[r, "\alpha^{u,v} X^k"] \arrow[d, "\mathcal{X}^u \mathcal{X}^v x^w"] \arrow[dr, phantom, "\mathsf{N}"] & \mathcal{X}^{u;v} X^k \arrow[d, "\mathcal{X}^{u;v} x^w"] \\
& \mathcal{X}^u \mathcal{X}^v \mathcal{X}^w X^l \arrow[r, "\alpha^{u,v} \mathcal{X}^w X^l"] \arrow[dl, "\mathcal{X}^u \alpha^{v,w} X^l"] \arrow[dr, phantom, "\mathsf{A}"] & \mathcal{X}^{u;v} \mathcal{X}^w X^l \arrow[d, "\alpha^{u;v,w} X^l"] \\
\mathcal{X}^u \mathcal{X}^{v;w} X^l \arrow[rr, "\alpha^{u,v;w} X^l"] & & \mathcal{X}^{u;v;w} X^l
\end{tikzcd}
$$

$\square$

Definition 5.3 Let Y^i be the colimit with cocone $z^u : \mathcal{X}^u X^j \to Y^i$, *i.e.*

$$
\begin{tikzcd}
\mathcal{X}^u X^j \arrow[r, "z^u"] \arrow[d, "\mathcal{X}^u x^v"'] \arrow[dr, phantom, "\mathsf{E}"] & Y^i \\
\mathcal{X}^u \mathcal{X}^v X^k \arrow[r, "\alpha^{u,v} X^k"] & \mathcal{X}^{u;v} X^k \arrow[u, "z^{u;v}"']
\end{tikzcd}
$$

commutes for all $u \in i/\mathcal{I}$.

Construction 5.4 An isomorphism $y^t : Y^{i'} \cong \mathcal{X}^t Y^i$.

Proof Let $t : i' \to i$ in $\mathcal{I}$. There is a diagram of type $i/\mathcal{I}$ given by applying $\mathcal{X}^t$ to the previous construction, and since this functor is continuous, $\mathcal{X}^t z^u$ is a colimiting cocone. Using ZA, $\alpha^{t,u} X^j \,;\, z^{t;u} : \mathcal{X}^t \mathcal{X}^u X^j \to Y^{i'}$ is a cocone, so there is a unique mediator $\mathcal{X}^t Y^i \to Y^{i'}$. Now $(\alpha^{t,u} X^j)^{-1} \,;\, \mathcal{X}^t z^u : \mathcal{X}^{t;u} X^j \to Y^i$ is a cocone over a final subdiagram of the diagram defining $Y^{i'}$ and so extends to a cocone over the whole diagram, whose mediator $y^t : Y^{i'} \to \mathcal{X}^t Y^i$ is the inverse. This is the unique map such that

$$
\begin{tikzcd}
Y^{i'} \arrow[r, "y^t"] \arrow[dr, phantom, "\mathsf{F}"] & \mathcal{X}^t Y^i \\
\mathcal{X}^{t;u} X^j \arrow[u, "z^{t;u}"] & \mathcal{X}^t \mathcal{X}^u X^j \arrow[l, "\alpha^{t,u} X^j"'] \arrow[u, "\mathcal{X}^t z^u"']
\end{tikzcd}
\qquad \text{over } j/\mathcal{I}
$$

commutes for all $u : i \to j$, the vertical maps forming colimiting cocones.

Lemma 5.5 $\langle Y^i, y^u \rangle$ is an object of $\mathsf{Lim}\, i : \mathcal{I}.\mathcal{X}^i \subset \Pi i : \mathcal{I}.\mathcal{X}^i$.

Proof For the U equation:

$$
\begin{tikzcd}
Y^i \arrow[r, "y^{\mathsf{id}} =?", "\gamma^i Y^i"'] & \mathcal{X}^{\mathsf{id}} Y^i \\
\mathcal{X}^u X^j \arrow[u, "z^u"] \arrow[r, shift right=2, "\gamma^i \mathcal{X}^u X^j"'] \arrow[ur, phantom, "\mathsf{N}"] & \mathcal{X}^{\mathsf{id}} \mathcal{X}^u X^j \arrow[l, shift right=2, "\alpha^{\mathsf{id},u} X^j"'] \arrow[u, "\mathcal{X}^{\mathsf{id}} z^u"']
\end{tikzcd}
$$

which is the colimit of U, and the associative law is obtained in the same way:

$$
\begin{tikzcd}
\mathcal{X}^u\mathcal{X}^vY^k \arrow[rrr, "\alpha^{u,v}Y^k"] & & & \mathcal{X}^{u;v}Y^k \\
& \mathcal{X}^u\mathcal{X}^v\mathcal{X}^wX^l \arrow[ul, "\mathcal{X}^u\mathcal{X}^vz^w"'] \arrow[r, "\alpha^{u,v}\mathcal{X}^wX^l"] \arrow[d, "\mathcal{X}^u\alpha^{u,v}X^l"] \arrow[r, phantom, bend left=60, "\mathsf{N}"] \arrow[dr, phantom, "\mathsf{A}"] & \mathcal{X}^{u;v}\mathcal{X}^wX^l \arrow[ur, "\mathcal{X}^{u;v}z^w"] \arrow[d, "\alpha^{u;v,w}X^l"'] & \\
& \mathcal{X}^u\mathcal{X}^{v;w}X^l \arrow[r, "\alpha^{u,v;w}X^l"] \arrow[dl, "\mathcal{X}^uz^{v;w}"] \arrow[r, phantom, bend right=60, "\mathsf{F}"] & \mathcal{X}^{u;v;w}X^l \arrow[dr, "z^{u;v;w}"] & \\
\mathcal{X}^uY^j \arrow[uuu, "\mathcal{X}^uy^v"] \arrow[uur, phantom, "\mathsf{F}"] & & & Y^i \arrow[lll, "y^u"] \arrow[uuu, "y^{u;v}"'] \arrow[uul, phantom, "\mathsf{F}"]
\end{tikzcd}
$$

□

Construction 5.6 Morphism $\zeta : \langle X^i, x^u\rangle \to \langle Y^i, y^u\rangle$ by $\zeta^i = \gamma^iX^i \,;\, z^{\mathsf{id}_i}$.

Proof It is a morphism:

$$
\begin{tikzcd}
X^i \arrow[r, "x^u"] \arrow[d, "\gamma^iX^i"'] & \mathcal{X}^uX^j \arrow[r, equal] \arrow[d, "\gamma^i\mathcal{X}^uX^j"] \arrow[dl, phantom, "\mathsf{Z}"] & \mathcal{X}^uX^j \arrow[r, equal] \arrow[d, "\mathsf{id}"] \arrow[dl, phantom, "\mathsf{L}"] & \mathcal{X}^uX^j \arrow[d, "\mathcal{X}^u\gamma^iX^j"] \arrow[dl, phantom, "\mathsf{R}"] \\
\mathcal{X}^{\mathsf{id}}X^i \arrow[r, "\mathcal{X}^{\mathsf{id}}x^u"] \arrow[d, "z^{\mathsf{id}_i}"'] & \mathcal{X}^{\mathsf{id}}\mathcal{X}^uX^j \arrow[r, "\alpha^{\mathsf{id},u}X^j"] \arrow[d, phantom, "\mathsf{E}"] & \mathcal{X}^uX^j \arrow[d, "z^u"'] & \mathcal{X}^u\mathcal{X}^{\mathsf{id}}X^j \arrow[l, "\alpha^{u,\mathsf{id}}X^j"'] \arrow[d, "\mathcal{X}^uz^{\mathsf{id}_j}"'] \\
Y^i \arrow[rr, equal] & & Y^i \arrow[r, "y^u"] & \mathcal{X}^uY^j
\end{tikzcd}
$$

□

Lemma 5.7 ζ is universal into the pseudolimit.

Proof Suppose $\langle b^i\rangle : \langle X^i, x^u\rangle \to \langle A^i, a^u\rangle$ is a morphism with a^u invertible, so the square

$$
\begin{tikzcd}
X^i \arrow[r, "b^i"] \arrow[d, "x^u"'] \arrow[dr, phantom, "\mathsf{H}"] & A^i \arrow[d, "a^u"] \\
\mathcal{X}^uX^j \arrow[r, "\mathcal{X}^ub^j"] & \mathcal{X}^uA^j
\end{tikzcd}
$$

commutes. I claim that $\mathcal{X}^u b^j\,;(a^u)^{-1}$ is a cocone for the diagram defining Y^i:

$$\begin{tikzcd}
\mathcal{X}^u X^j \arrow[r, "\mathcal{X}^u b^j"] \arrow[d, "\mathcal{X}^u x^u"'] \arrow[dr, phantom, "\mathsf{H}"] & \mathcal{X}^u A^j \arrow[d, "\mathcal{X}^u a^v"'] & A^i \arrow[l, "a^u"'] \arrow[dd, equal] \arrow[ddl, phantom, "\mathsf{M}"] \\
\mathcal{X}^u\mathcal{X}^v X^k \arrow[r, "\mathcal{X}^u\mathcal{X}^v b^k"] \arrow[d, "\alpha^{u,v} X^k"'] \arrow[dr, phantom, "\mathsf{Z}"] & \mathcal{X}^u\mathcal{X}^v A^k \arrow[d, "\alpha^{u,v} A^k"'] & \\
\mathcal{X}^{u;v} X^k \arrow[r, "\mathcal{X}^{u;v} b^k"] & \mathcal{X}^{u;v} A^k & A^i \arrow[l, "a^{u;v}"']
\end{tikzcd}$$

Hence there is a unique mediator $c^i : Y^i \to A^i$ such that

$$\begin{tikzcd}
Y^i \arrow[r, "c^i"] \arrow[dr, phantom, "\mathsf{G}"] & A^i \arrow[d, "a^u"] \\
\mathcal{X}^u X^j \arrow[u, "z^u"] \arrow[r, "\mathcal{X}^u b^j"] & \mathcal{X}^u A^j
\end{tikzcd}$$

commutes. Finally the following diagram shows that $c : Y \to A$ is a morphism:

$$\begin{tikzcd}
Y^i \arrow[rrr, "c^i"] \arrow[ddd, "y^u"'] & & \arrow[d, phantom, "\mathsf{G}"] & A^i \arrow[d, "a^u"] \\
\mathsf{F} & \mathcal{X}^u X^j \arrow[ul, "z^u"'] \arrow[rr, "\mathcal{X}^u b^j"] \arrow[d, shift left, "\mathcal{X}^u\gamma^j X^j"] \arrow[d, phantom, "\mathsf{U}"] & \arrow[d, phantom, "\mathsf{Z}"] & \mathcal{X}^u A^j \arrow[d, "\mathcal{X}^u\gamma^j A^j"] \\
& \mathcal{X}^u\mathcal{X}^i dX^j \arrow[u, shift left, "\alpha^{u,\mathsf{id}} X^j"] \arrow[rr, "\mathcal{X}^u\mathcal{X}^{\mathsf{id}} b^j"] \arrow[dl, "\mathcal{X}^u z^{\mathsf{id}}"'] & \arrow[d, phantom, "\mathsf{G}"] & \mathcal{X}^u\mathcal{X}^{\mathsf{id}} A^j \\
\mathcal{X}^u Y^j \arrow[rrr, "\mathcal{X}^u c^j"] & & & \mathcal{X}^u A^j \arrow[u, "\mathcal{X}^u a^{\mathsf{id}}", "\mathcal{X}^u\gamma^j A^j"']
\end{tikzcd}$$

□

Proposition 5.8 The full inclusion functor $\mathsf{Lim}\, i : \mathcal{I}.\mathcal{X}^i \subset \Pi i : \mathcal{I}.\mathcal{X}^i$ is continuous and has a reflection (left adjoint post-inverse).

Proof The first part is immediate from Proposition 5.1; note also that the inclusion is full because the two categories have the same definition of morphisms. We have just found the unit of the adjunction; observe that if $\vec{x}$ are already isomorphisms then ζ is also an isomorphism. □

There is an application of these results to finding filtered colimits in Σ-types.

Definition 5.9 A functor $P : \mathcal{S} \to \mathcal{C}$ is a *continuous fibration* if

(i) it is a fibration,

(ii) the fibres $\mathcal{X}^i$ have, and the functors $\mathcal{X}^u$ preserve, filtered colimits (of types $\mathcal{J}$),

(iii) $\mathcal{C}$ has filtered colimits (of types $\mathcal{I}$), and

(iv) if $c_i : C_i \to C$ is a colimiting cocone for a diagram c_u of type $\mathcal{I}$ then $\mathcal{X}^{c_i}$ is a pseudolimiting cone.

Lemma 5.10 An object over a colimit is the colimit of its liftings.

Proof Let $c_i : C_i \to C$ be a colimiting cocone in $\mathcal{C}$ and $X \in \mathcal{X}^C$. Let $v_{c_i}X : \langle C_i, X^i\rangle \to \langle C, X\rangle$ be the horizontal lifting of c_i, where $X^i = \mathcal{X}^{c_i}X$; then with $x^u = v_{c_u}$ we have an object of $\Pi i : \mathcal{I}.\mathcal{X}^{C_i}$. By (iv) this corresponds to an object $Y \in \mathcal{X}^C$ with $X^i \cong \mathcal{X}^{c_i}Y$... □

Proposition 5.11 If $P : \mathcal{S} \to \mathcal{C}$ is a continuous fibration then $\mathcal{S}$ has and P preserves filtered colimits (of type $\mathcal{I}$).

Proof We are given $\mathcal{I} \to \mathcal{S}$, a diagram of type $\mathcal{I}$, and suppose that the fibration P corresponds to a pseudofunctor $\mathcal{C}^{\mathrm{op}} \to \mathbf{Cat}$. Write

$$C^{(-)} : \mathcal{I} \to \mathcal{S} \to \mathcal{C} \quad \text{and} \quad \mathcal{X}^{(-)} : \mathcal{I}^{\mathrm{op}} \to \mathcal{S}^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}} \to \mathbf{Cat}$$

so that the diagram has vertices $X_i \in \mathcal{X}^i$. The arrows of the diagram are of the form $x^u : X^i \to \mathcal{X}^u X^j$ and satisfy the *equations, and so we have an object of $\Pi i : \mathcal{I}.\mathcal{X}^i$. Applying the reflector yields an object $\langle Y^i, y^u\rangle$ of $\mathsf{Lim}\, i : \mathcal{I}.\mathcal{X}^i$, which corresponds to a unique object Y (up to isomorphism) of the fibre over $C = \mathsf{colim}^{\uparrow} C_i$. By the lemma, Y is the colimit of $\vec{Y}$, and from the reflection it is also the colimit of the $\vec{X}$. (explain why) □

Every object is the colimit of its liftings; the colimiting cocone of a diagram of horizontal maps is horizontal.

6 Adjunctions

Recall that we call two functors $h : \mathcal{A} \to \mathcal{B}$, $c : \mathcal{B} \to \mathcal{A}$ *adjoint*, and write $c \dashv h$, if there is a bijection

$$\frac{cB \to \ \ A}{B \to hA}$$

which is natural in A and B. In particular, corresponding to $\mathsf{id} : cB \to cB$ there is the *unit* $\eta B : B \to hcB$ and to $\mathsf{id} : hA \to hA$ the *counit* $\epsilon A : chA \to A$. These are natural in A and satisfy the *triangular identities*:

$$\begin{tikzcd}
c \arrow[r, "c\eta"] \arrow[dr, "\mathsf{id}"'] & chc \arrow[d, "\epsilon c"] \arrow[dl, phantom, "\dashv"] \\
 & c
\end{tikzcd}
\qquad
\begin{tikzcd}
hch \arrow[d, "c\epsilon"'] \arrow[dr, phantom, "\dashv"] & h \arrow[l, "\eta h"'] \arrow[dl, "\mathsf{id}"] \\
h &
\end{tikzcd}$$

[Mac Lane 1971], §4.1 shows that (c, h, η, ϵ) suffice to characterise the adjunction. Since we no longer have any mention of the objects of the categories $\mathcal{A}$ and $\mathcal{B}$, the latter may now be objects of an abstract 2-category $\mathbf{C}$, and we have an equational *definition* of an adjunction between two opposite arrows (1-cells) between two objects (0-cells) of a 2-category.

The above natural bijection is a well-known and easy to remember way of presenting the adjunction, but it has a less well known dual. Instead of applying c and h to *objects* of (or functors *into*) $\mathcal{A}$ and $\mathcal{B}$, we can apply functors *out of* these categories:

$$\frac{\hat{B}h \to \hat{A}}{\hat{B} \ \ \to \hat{A}c}$$

naturally in $\hat{A} : \mathcal{A} \to \mathcal{X}$ and $\hat{B} : \mathcal{B} \to \mathcal{X}$. We could say "$h$ is co-left adjoint to c."

Construction 6.1 A natural bijection between natural transformations $\theta : h_1 \to h_2$ and $\phi : c_2 \to c_1$ which identifies identities. Using both correspondences together we obtain

$$\begin{tikzcd}
 & & h_2c_2h_1 \arrow[ddr, "h_2\kappa"] & h_1 \arrow[l, "\eta_2 h_1"'] \arrow[r, "\iota h_1"] \arrow[dd, "\theta"] & h_2c_1h_1 \arrow[ddl, "h_2\epsilon_1"'] & & \\
c_2h_2 \arrow[ddrr, "\epsilon_2"] & & & & & & h_1c_1 \arrow[ddll, "\theta c_1"'] \\
 & & & h_2 & & & \\
c_2h_1 \arrow[uu, "c_2\theta"] \arrow[rr, "\kappa"] \arrow[dd, "\phi h_1"'] & & \mathrm{id} & & h_2c_1 & & \mathrm{id} \arrow[uu, "\eta_1"'] \arrow[ll, "\iota"'] \arrow[dd, "\eta_2"] \\
 & & & c_1 & & & \\
c_1h_1 \arrow[uurr, "\epsilon_1"'] & & & & & & h_2c_2 \arrow[uull, "h_2\phi"] \\
 & & c_2h_1c_1 \arrow[uur, "\kappa c_1"] & c_2 \arrow[l, "c_2\eta_1"'] \arrow[r, "c_2\iota"] \arrow[uu, "\phi"] & c_2h_2c_1 \arrow[uul, "\epsilon_2 c_1"'] & &
\end{tikzcd}$$

(commutation of one of which suffices) where we can verify $\theta \mapsto \iota \mapsto \phi \mapsto \kappa \mapsto \theta$ are bijections using the triangle laws. $\square$

We shall write

$$\begin{tikzcd}
\mathrm{id} \arrow[r, "\eta_1"] \arrow[d, "\eta_2"'] \arrow[dr, phantom, "\mathsf{D}"] & h_1c_1 \arrow[d, "\theta c_1"] \\
h_2c_2 \arrow[r, "h_2\phi"] & h_2c_1
\end{tikzcd}$$

Lemma 6.2 The bijection respects composition and identites.

Proof It suffices to show that $\theta\,;\theta'$ and $\phi'\,;\phi$ satisfy the equation.

$$\begin{tikzcd}
\mathrm{id} \arrow[rr, "\eta_1"] \arrow[dr, "\eta_2"] \arrow[dd, "\eta_3"'] & \arrow[d, phantom, "\mathsf{D}"] & h_1c_1 \arrow[d, "\theta c_1"] \\
\arrow[r, phantom, "\mathsf{D}'"] & h_2c_2 \arrow[r, "h_2\phi"] \arrow[d, "\theta' c_2"'] \arrow[dr, phantom, "\mathsf{Z}"] & h_2c_1 \arrow[d, "\theta' c_1"] \\
h_3c_3 \arrow[r, "h_3\phi'"] & h_3c_2 \arrow[r, "h_3\phi"] & h_3c_1
\end{tikzcd}$$

The result for identities is obvious. $\square$

Definition 6.3 The 2-category $\mathbf{C}^{\dashv}$ has the same objects as $\mathbf{C}$; its 1-cells are adjunctions (c, h, η, ϵ) and its 2-cells are pairs of natural transformations (ϕ, θ) satisfying the equations of the construction. The composition is in the obvious way.

Proposition 6.4 The forgetful 2-functor $\mathbf{C}^{\dashv} \to \mathbf{C}$ which extracts the left (or right) part of the adjunction and natural transformations is full and faithful at the 2-level. Loosely speaking, the 2-categories of left and right adjoints are dual. $\square$

Note carefully that we shall find it convenient to write composition in $\mathcal{I}$ both right-handedly as $u\,;v$ and left-handedly $v \circ u$: *these notations are completely synonymous.* We shall also adopt the convention of writing $\mathcal{X}^u$ for the functors in the limit diagram (which are *contravariant* in u) and $\mathcal{X}_u$ for their left adjoints in the colimit diagram (which are *covariant* in u).

Definition 6.5 Let $P : \mathcal{S} \to \mathcal{C}$ be a functor. A morphism $f : X \to Y$ is *op-horizontal* over $Pf = u : i \to j$ if for any $h : X \to Z$ over $u\,;v : i \to j \to k$ there is a unique $g : Y \to Z$ over v with $h = f\,;g$. We call P an *op-fibration* if each u has an op-horizontal lifting at each X with $PX = i$. If P is both a fibration and an op-fibration we call it a *bifibration.*

Lemma 6.6 $P : \mathcal{S} \to \mathcal{C}$ is an op-fibration iff $P : \mathcal{S}^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$ is a fibration.

Proof Obvious. $\square$

Proposition 6.7 Let $\mathcal{X}^{(-)} : \mathcal{I}^{\mathrm{op}} \to \mathbf{Cat}$ be a pseudofunctor. Then $P : \Sigma i : \mathcal{I}.X^i$ is a bifibration iff each $\mathcal{X}^u$ has a left adjoint, written $\mathcal{X}_u$.

Note 6.8 Since the left adjoints (op-substitutions) $\mathcal{X}_u$ are covariant in $u : i \to j$, but we still write composition of functors from right to left, it is convenient to use the notations

$$u\,;v \quad \text{and} \quad v \circ u$$

synonymously for the composite $i \xrightarrow{u} j \xrightarrow{v} k$.

Proof

[⇒] Any morphism $(u, f) : (i, X) \to (j, Y)$ can be factorised as a vertical followed by a horizontal, or as an op-horizontal followed by a vertical:

$$\begin{tikzcd}
(i,X) \arrow[r] \arrow[d] \arrow[dr, "{(u,f)}"] & (j, \mathcal{X}_u X) \arrow[d] \\
(i, \mathcal{X}^u Y) \arrow[r] & (j, Y)
\end{tikzcd}$$

where $\mathcal{X}_u X$ is introduced as the codomain of the op-horizontal lifting. Making such a choice for all $X \in \mathcal{X}^i$, we can extend this uniquely to a functor $\mathcal{X}_u : \mathcal{X}^i \to \mathcal{X}^j$. I claim this is the left adjoint: f corresponds bijectively to either vertical part $X \to \mathcal{X}^u Y$ or $\mathcal{X}_u X \to Y$. The unit and counit are given by

$$\begin{tikzcd}
X \arrow[d, "\eta^u X"'] \arrow[dr, "\text{ophoriz}"] & \\
\mathcal{X}^u \mathcal{X}_u X \arrow[r, "\text{horiz}"] & \mathcal{X}_u X
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
& \mathcal{X}_u \mathcal{X}^u Y \arrow[d, "\epsilon_u Y"] \\
\mathcal{X}^u Y \arrow[ur, "\text{ophoriz}"] \arrow[r, "\text{horiz}"] & Y
\end{tikzcd}$$

and the equations $\eta^u \mathcal{X}^u\,; \mathcal{X}^u \epsilon_u = \mathsf{id}$ and $\mathcal{X}_u \eta^u\,; \epsilon_u \mathcal{X}_u = \mathsf{id}$ may be verified using the universal properties of the horizontal map $\mathcal{X}^u Y \to Y$ and the op-horizontal map $X \to \mathcal{X}_u X$ respectively.

[⇐] I claim that $(u, \eta^u X) : (i, X) \to (j, \mathcal{X}_u X)$ is op-horizontal. Let $(u\,; v, f) : (i, X) \to (k, Z)$, so that $f' = f\,; (\alpha^{u,v} Z)^{-1} : X \to \mathcal{X}^u \mathcal{X}^v Z$ in $\mathcal{X}^i$. Let $f'' : \mathcal{X}_u X \to \mathcal{X}^v Z$ be its adjoint transpose; this is the unique solution of $f' = \eta^u X\,; \mathcal{X}^u f''$. Then $(v, f'') : (j, \mathcal{X}_u X) \to (k, Z)$ is the required factorisation. $\square$

Lemma 6.9 The functors $\mathcal{X}_u$ have coherences $\delta_i : \mathcal{X}_{\mathsf{id}} \to \mathsf{id}$ and $\beta_{v,u} : \mathcal{X}_{v \circ u} \to \mathcal{X}_v \mathcal{X}_u$ which satisfy equations dual to those in definition 2.1:

$$\begin{tikzcd}
\mathcal{X}_w \mathcal{X}_v \mathcal{X}_u & \mathcal{X}_w \mathcal{X}_{v\circ u} \arrow[l, "\mathcal{X}_w \beta_{v,u}"'] \\
\mathcal{X}_{w\circ v} \mathcal{X}_u \arrow[u, "\beta_{w,v}\mathcal{X}_u"] \arrow[ur, phantom, "\mathsf{A}"] & \mathcal{X}_{w\circ v\circ u} \arrow[l, "\beta_{w\circ v,u}"'] \arrow[u, "\beta_{w,v\circ u}"']
\end{tikzcd} \qquad \text{associativity}$$

and

$$\begin{tikzcd}
\mathcal{X}_u & \mathcal{X}_{\mathsf{id}_j} \mathcal{X}_u \arrow[l, "\delta_j \mathcal{X}_u"'] \\
& \mathcal{X}_u \arrow[u, "\beta_{\mathsf{id}_j,u}"'] \arrow[ul, "\mathsf{id}\mathcal{X}_u", "\mathsf{L}"']
\end{tikzcd} \qquad
\begin{tikzcd}
\mathcal{X}_u \mathcal{X}_{\mathsf{id}_i} \arrow[r, "\mathcal{X}_u \delta_i"] & \mathcal{X}_u \\
\mathcal{X}_u \arrow[u, "\beta_{u,\mathsf{id}_i}"] \arrow[ur, "\mathsf{R}", "\mathsf{id}\mathcal{X}_u"'] &
\end{tikzcd} \qquad \text{unit}$$

Proof Immediate from lemmas 3.7 and 6.8. $\square$

Lemma 6.10 The coherences $\alpha^{u,v}$, $\beta_{v,u}$, γ^i and δ_i are related to the units by the *equations

$$\begin{tikzcd}
\mathsf{id} \arrow[r, "\eta^{\mathsf{id}}"] \arrow[d, "\mathsf{id}"'] \arrow[dr, phantom, "\mathsf{C}"] & \mathcal{X}^{\mathsf{id}} \mathcal{X}_{\mathsf{id}} \arrow[d, "\mathcal{X}^{\mathsf{id}} \delta_k"] \\
\mathsf{id} \arrow[r, "\gamma^k"] & \mathcal{X}^{\mathsf{id}}
\end{tikzcd} \quad \text{and} \quad
\begin{tikzcd}
\mathsf{id} \arrow[r, "\eta^w"] \arrow[d, "\eta^{w;w'}"'] \arrow[drr, phantom, "\mathsf{B}"] & \mathcal{X}^w \mathcal{X}_w \arrow[r, "\mathcal{X}^w \eta^{w'} \mathcal{X}_w"] & \mathcal{X}^w \mathcal{X}^{w'} \mathcal{X}_{w'} \mathcal{X}_w \arrow[d, "\alpha^{w,w'} \mathcal{X}_{w'} \mathcal{X}_w"'] \\
\mathcal{X}^{w;w'} \mathcal{X}_{w' \circ w} \arrow[rr, "\mathcal{X}^{w;w'} \beta_{w',w}"] & & \mathcal{X}^{w;w'} \mathcal{X}_{w'} \mathcal{X}_w
\end{tikzcd}$$

and to the counits by the equations

$$\begin{tikzcd}
\mathsf{id} & \mathcal{X}_{\mathsf{id}} \mathcal{X}^{\mathsf{id}} \arrow[l, "\epsilon_{\mathsf{id}}"'] \\
\mathsf{id} \arrow[u, "\mathsf{id}"] \arrow[ur, phantom, "\mathsf{C}"] & \mathcal{X}_{\mathsf{id}} \arrow[l, "\delta_k"'] \arrow[u, "\mathcal{X}_{\mathsf{id}} \gamma^k"']
\end{tikzcd} \quad \text{and} \quad
\begin{tikzcd}
\mathsf{id} & \mathcal{X}_{w'} \mathcal{X}^{w'} \arrow[l, "\epsilon_{w'}"'] & \mathcal{X}_{w'} \mathcal{X}_w \mathcal{X}^w \mathcal{X}^{w'} \arrow[l, "\mathcal{X}_{w'} \epsilon_w \mathcal{X}^{w'}"'] \\
\mathcal{X}_{w' \circ w} \mathcal{X}^{w;w'} \arrow[u, "\epsilon_{w' \circ w}"] \arrow[urr, phantom, "\mathsf{B}"] & & \mathcal{X}_{w'} \mathcal{X}_w \mathcal{X}^{w;w'} \arrow[ll, "\beta_{w',w} \mathcal{X}^{w;w'}"'] \arrow[u, "\mathcal{X}_w \mathcal{X}_{w'} \alpha^{w,w'}"']
\end{tikzcd}$$

$\square$

7 Transfer functors

Now we shall begin the construction of the equivalence between the pseudolimit and pseudocolimit. Imagining for the moment that we already have this equivalence, we can form

$$\mathcal{X}^i \xrightarrow{\mathcal{X}_i} \operatorname*{\mathsf{Colim}}_{k \in \mathcal{I}} \mathcal{X}^k \sim \operatorname*{\mathsf{Lim}}_{k \in \mathcal{I}} \mathcal{X}^k \xrightarrow{\mathcal{X}^j} \mathcal{X}^j$$

using the colimiting cocone and limiting cone. It is our purpose now to construct this composite functor, M_i^j. When we have it we can use the universal property of the pseudolimit to construct functors $\mathcal{X}^i \to \mathsf{Lim}\, k : \mathcal{I}.\mathcal{X}^k$; these will be left adjoints to $\mathcal{X}^i$ and will form the colimiting cocone $\mathcal{X}_i$.

Construction 7.1 A Diagram of type $(i,j)/\mathcal{I}$ of functors $\mathcal{X}^i \to \mathcal{X}^j$. On objects, $(i \xrightarrow{u} k \xleftarrow{v} j) \mapsto \mathcal{X}^v\mathcal{X}_u$. The image of $w : k \to k'$ is

$$
\begin{tikzcd}
\mathcal{X}^v\mathcal{X}_u \arrow[r, "\mathcal{X}^v\eta^w\mathcal{X}_u"] & \mathcal{X}^v\mathcal{X}^w\mathcal{X}_w\mathcal{X}_u \arrow[r, "\alpha^{v,w}\mathcal{X}_w\mathcal{X}_u"] \arrow[dr, phantom, "\mathsf{N}"] & \mathcal{X}^{v;w}\mathcal{X}_w\mathcal{X}_u \\
& \mathcal{X}^v\mathcal{X}^w\mathcal{X}_{w\circ u} \arrow[u, "\mathcal{X}^v\mathcal{X}^w\beta_{w,u}"] \arrow[r, "\alpha^{v,w}\mathcal{X}_{w\circ u}"'] & \mathcal{X}^{v;w}\mathcal{X}_{w\circ u} = \mathcal{X}^{v'}\mathcal{X}_{u'} \arrow[u, "\mathcal{X}^{v;w}\beta_{w,u}"']
\end{tikzcd}
$$

Proof We have to show that we have defined a functor. The effect on the identity id_k is the long anticlockwise route from $\mathcal{X}^v\mathcal{X}_u$ on the top left to $\mathcal{X}^v\mathcal{X}_u$ on the top right. We use the correspondence between γ^k and δ_k, together with one unit and one counit law.

$$
\begin{tikzcd}
\mathcal{X}^v\mathcal{X}_u \arrow[r, "\mathsf{id}"] \arrow[d, "\eta^{\mathsf{id}}"'] \arrow[dr, phantom, "\mathsf{C}"] & \mathcal{X}^v\mathcal{X}_u \arrow[r, "\mathsf{id}"] \arrow[d, "\gamma^k"'] \arrow[r, phantom, bend right, "\mathsf{R}"] & \mathcal{X}^v\mathcal{X}_u \\
\mathcal{X}^v\mathcal{X}^{\mathsf{id}}\mathcal{X}_{\mathsf{id}}\mathcal{X}_u \arrow[r, "\delta_k"] & \mathcal{X}^v\mathcal{X}^{\mathsf{id}}\mathcal{X}_u \arrow[ur, "\alpha^{v,\mathsf{id}}"'] & \\
\mathcal{X}^v\mathcal{X}^{\mathsf{id}}\mathcal{X}_u \arrow[u, "\beta_{\mathsf{id},u}"] \arrow[ur, "\mathsf{id}"'] \arrow[ur, phantom, bend left, "\mathsf{R}"] & &
\end{tikzcd}
$$

For the composite $w\,;w'$,

$$
\begin{tikzcd}
\mathcal{X}^v\mathcal{X}_u \arrow[rrr, equal] \arrow[d, "\eta^w"] & & & \mathcal{X}^v\mathcal{X}_u \arrow[d, "\eta^{w;w'}"'] \\
\mathcal{X}^v\mathcal{X}^w\mathcal{X}_w\mathcal{X}_u \arrow[r, "\eta^{w'}"] \arrow[d, "\alpha^{v,w}"] \arrow[dr, phantom, "\mathsf{Z}"] & \mathcal{X}^v\mathcal{X}^w\mathcal{X}^{w'}\mathcal{X}_{w'}\mathcal{X}_w\mathcal{X}_u \arrow[r, "\alpha^{w,w'}"] \arrow[d, "\alpha^{v,w}"'] \arrow[dr, phantom, "\mathsf{A}"] \arrow[u, phantom, "\mathsf{B}"] & \mathcal{X}^v\mathcal{X}^{w;w'}\mathcal{X}_{w'}\mathcal{X}_w\mathcal{X}_u \arrow[d, "\alpha^{v,w;w'}"'] \arrow[dr, phantom, "\mathsf{Z}"] & \mathcal{X}^v\mathcal{X}^{w;w'}\mathcal{X}_{w'\circ w}\mathcal{X}_u \arrow[l, "\beta_{w',w}"'] \arrow[d, "\alpha^{v,w;w'}"] \\
\mathcal{X}^{v;w}\mathcal{X}_w\mathcal{X}_u \arrow[r, "\eta^{w'}"] \arrow[dr, phantom, "\mathsf{N}"] & \mathcal{X}^{v;w}\mathcal{X}^{w'}\mathcal{X}_{w'}\mathcal{X}_w\mathcal{X}_u \arrow[r, "\alpha^{v;w,w'}"] \arrow[dr, phantom, "\mathsf{N}"] & \mathcal{X}^{v;w;w'}\mathcal{X}_{w'}\mathcal{X}_w\mathcal{X}_u \arrow[dr, phantom, "\mathsf{A}"] & \mathcal{X}^{v;w;w'}\mathcal{X}_{w'\circ w}\mathcal{X}_u \arrow[l, "\beta_{w',w}"'] \\
\mathcal{X}^{v;w}\mathcal{X}_{w\circ u} \arrow[r, "\eta^{w'}"] \arrow[u, "\beta_{w,u}"'] & \mathcal{X}^{v;w}\mathcal{X}^{w'}\mathcal{X}_{w'}\mathcal{X}_{w\circ u} \arrow[r, "\alpha^{v;w,w'}"] \arrow[u, "\beta_{w,u}"] & \mathcal{X}^{v;w;w'}\mathcal{X}_{w'}\mathcal{X}_{w\circ u} \arrow[u, "\beta_{w,u}"] & \mathcal{X}^{v;w;w'}\mathcal{X}_{w'\circ v\circ u} \arrow[l, "\beta_{w',w\circ u}"'] \arrow[u, "\beta_{w'\circ w,u}"]
\end{tikzcd}
$$

where the left-hand side is the effect on w, the bottom that on w' and the right-hand side that on $w\,;w'$. $\square$

Construction 7.2 Functor $M_i^j : \mathcal{X}^i \to \mathcal{X}^j$ is the (pointwise) colimit, with cocone $\nu_u^v : \mathcal{X}^v\mathcal{X}_u \to$

M^j_i, *i.e.*

$$\begin{tikzcd}
\mathcal{X}^v\mathcal{X}_u \arrow[rr, "\nu^v_u"] \arrow[d, "\mathcal{X}^v\eta^w\mathcal{X}_u"'] \arrow[drr, phantom, "\mathsf{K}"] & & M^j_i \\
\mathcal{X}^v\mathcal{X}^w\mathcal{X}_w\mathcal{X}_u \arrow[r, "\alpha^{v,w}\mathcal{X}_w\mathcal{X}_u"] & \mathcal{X}^{v;w}\mathcal{X}_w\mathcal{X}_u & \mathcal{X}^{v;w}\mathcal{X}_{w\circ u} \arrow[l, "\mathcal{X}^{v;w}\beta_{w,u}"'] \arrow[u, "\nu^{v;w}_{w\circ u}"']
\end{tikzcd}$$

Proof We know that $(i,j)/\mathcal{I}$ is a filtered category, so it is possible to form the colimit M^i_jX for each X; it is standard that this extends to a functor which is the colimit in the functor category. In detail, for $f : X \to Y$ we have a cocone $\mathcal{X}^v\mathcal{X}_uf\,;\nu^v_uY : \mathcal{X}^v\mathcal{X}_uX \to M^i_jY$ by naturality of η^w, $\alpha^{v,w}$ and $\beta_{w,u}$, so M^i_jf is the unique mediator and it is automatic that ν^v_u is natural. □

Construction 7.3 Coherences $\mu^t_s : \mathcal{X}^tM^j_i\mathcal{X}_s \cong M^{j'}_{i'}$ for $s : i' \to i$ and $t : j' \to j$ such that

$$\begin{tikzcd}
\mathcal{X}^tM^j_i\mathcal{X}_s \arrow[rr, "\mu^t_s"] \arrow[drr, phantom, "\mathsf{P}"] & & M^{j'}_{i'} \\
\mathcal{X}^t\mathcal{X}^v\mathcal{X}_u\mathcal{X}_s \arrow[u, "\mathcal{X}^t\nu^v_u\mathcal{X}_s"] \arrow[r, "\alpha^{t,v}\mathcal{X}_u\mathcal{X}_s"] & \mathcal{X}^{t,v}\mathcal{X}_u\mathcal{X}_s & \mathcal{X}^{t;v}\mathcal{X}_{u\circ s} \arrow[l, "\mathcal{X}^{t;v}\beta_{u,s}"'] \arrow[u, "\nu^{t;v}_{u;s}"']
\end{tikzcd}$$

Proof The lower composite $\mathcal{X}^t\mathcal{X}^v\mathcal{X}_u\mathcal{X}_s \to M^{j'}_{i'}$ is a cocone by naturality of α, β and η and associativity of α and β (the proof is very similar to the diagram in 7.1). Since $\mathcal{X}^t$ is continuous, $\mathcal{X}^t\nu^v_u\mathcal{X}_s$ is colimiting and μ^t_s is then defined as the mediator. It is invertible because $(i,j)/\mathcal{I} \to (i',j')/\mathcal{I}$ is final, and natural by the universal property of the colimit. □

Lemma 7.4 The coherences $\mu^t_i = \mathcal{X}^tM^j_i\delta_i\,;\mu^t_{id_i} : \mathcal{X}^tM^j_i \cong M^{j'}_{i'}$ for $s : i' \to i$ satisfy U, M and

$$\begin{tikzcd}
\mathcal{X}^tM^j_i \arrow[r, "\mu^t_i"] \arrow[dr, phantom, "\mathsf{P}"] & M^{j'}_{i'} \\
\mathcal{X}^t\mathcal{X}^v\mathcal{X}_u \arrow[u, "\mathcal{X}^t\nu^v_u\mathcal{X}_s"] \arrow[r, "\alpha^{t,v}\mathcal{X}_u"] & \mathcal{X}^{t,v}\mathcal{X}_u \arrow[u, "\nu^{t;v}_{u;s}"']
\end{tikzcd}$$

Proof These are just $\mathsf{colim}^\uparrow\,\mathsf{U}\mathcal{X}_u$ and $\mathsf{colim}^\uparrow\,\mathsf{M}\mathcal{X}_u$. □

Definition 7.5 $\mathcal{X}_i : \mathcal{X}^i \to \mathsf{Lim}\,k : \mathcal{I}.\mathcal{X}^k$ such that $\mathcal{X}^j\mathcal{X}_i = M^j_i$ and $\xi^u\mathcal{X}_i = \mu^u_i$. We use the coherences μ^t_i and the universal property of the pseudolimit. □

8 Unit and counit

Now we shall show that $\mathcal{X}_i \dashv \mathcal{X}^i$ and that $\mathcal{X}_i$ is a colimiting cocone.

Lemma 8.1 The maps

$$\mathcal{X}^v\mathcal{X}_u\mathcal{X}^i \xrightarrow[\cong]{\mathcal{X}^v\mathcal{X}_u\xi^u} \mathcal{X}^v\mathcal{X}_u\mathcal{X}^u\mathcal{X}^k \xrightarrow{\mathcal{X}^v\epsilon_u\mathcal{X}^k} \mathcal{X}^v\mathcal{X}^k \xleftarrow[\cong]{\xi^v} \mathcal{X}^j$$

form a cocone for the diagram defining $M^j_i\mathcal{X}^i$.

Proof Recall that $\mathcal{X}^{a,b,c}_{d,e,f} \stackrel{\text{def}}{=} \mathcal{X}^a\mathcal{X}^b\mathcal{X}^c\mathcal{X}_d\mathcal{X}_e\mathcal{X}_f$, and we also omit the functors applied to natural transformations. Then the diagram

$$\begin{tikzcd}
\mathcal{X}^v_u\mathcal{X}^i \arrow[r, "\xi^u"] \arrow[d, "\eta^w"'] \arrow[dr, phantom, "\mathsf{Z}"] & \mathcal{X}^v_u\mathcal{X}^{u,k} \arrow[rr, "\epsilon_u"] \arrow[d, "\eta^w"'] \arrow[dr, phantom, "\mathsf{Z}"] & & \mathcal{X}^{v,k} \arrow[dl, "\eta^w"'] \arrow[dr, "\xi^w"] \arrow[dd, phantom, "\mathsf{N}"] & & \mathcal{X}^j \arrow[ll, "\xi^v"'] \arrow[dddd, equal] \\
\mathcal{X}^{v,w}_{w,u}\mathcal{X}^i \arrow[r, "\xi^u"] \arrow[dr, "\xi^{u;w}"'] & \mathcal{X}^{v,w}_{w,u}\mathcal{X}^{u,k} \arrow[r, "\epsilon_u"] \arrow[dr, "\xi^w"'] \arrow[d, phantom, "\mathsf{M}"] & \mathcal{X}^{v,w}_w\mathcal{X}^k \arrow[dr, "\xi^w"'] \arrow[d, phantom, "\mathsf{N}"] & & \mathcal{X}^{v,w,l} \arrow[dl, "\eta^w"'] \arrow[dd] \arrow[d, phantom, "\mathsf{id}"] \arrow[dr, phantom, "\mathsf{M}"] & \\
 & \mathcal{X}^{v,w}_{w,u}\mathcal{X}^{u;w,l} \arrow[dl, phantom, "\mathsf{Z}"] & \mathcal{X}^{v,w}_{w,u}\mathcal{X}^{u,w,l} \arrow[l, "\alpha^{u,w}"'] \arrow[r, "\epsilon_u"] & \mathcal{X}^{v,w}_w\mathcal{X}^{w,l} \arrow[dr, "\epsilon_w"] \arrow[r, phantom, "\mathsf{Y}"] \arrow[dl, phantom, "\mathsf{B}"] & & \\
\mathcal{X}^{v,w}_{w\circ u}\mathcal{X}^i \arrow[uu, "\beta_{w,u}"] \arrow[rr, "\xi^{u;w}"] \arrow[d, "\alpha^{v,w}"'] \arrow[drr, phantom, "\mathsf{Z}"] & & \mathcal{X}^{v,w}_{w\circ u}\mathcal{X}^{u;w,l} \arrow[ul, "\beta_{w,u}"'] \arrow[rr, "\epsilon_{w\circ u}"] \arrow[d, "\alpha^{v,w}"'] \arrow[drr, phantom, "\mathsf{Z}"] & & \mathcal{X}^{v,w,l} \arrow[d, "\alpha^{v,w}"'] & \\
\mathcal{X}^{v;w}_{w\circ u}\mathcal{X}^i \arrow[rr, "\xi^{u;w}"] & & \mathcal{X}^{v;w}_{w\circ u}\mathcal{X}^{u;w,l} \arrow[rr, "\epsilon_{w\circ u}"] & & \mathcal{X}^{v;w,l} & \mathcal{X}^j \arrow[l, "\xi^{v;w}"']
\end{tikzcd}$$

commutes. $\square$

Definition 8.2 Natural transformation $\epsilon^j_i : M^j_i\mathcal{X}^i \to \mathcal{X}^j$ is the mediator, the unique map such that

$$\begin{tikzcd}
M^j_i\mathcal{X}^i \arrow[rr, "\epsilon^j_i"] & & \mathcal{X}^j \arrow[d, "\xi^v"] \\
\mathcal{X}^v\mathcal{X}_u\mathcal{X}^i \arrow[u, "\nu^v_u\mathcal{X}^i"] \arrow[r, "\mathcal{X}^v\mathcal{X}_u\xi^u"] \arrow[urr, phantom, "\mathsf{Q}"] & \mathcal{X}^v\mathcal{X}_u\mathcal{X}^u\mathcal{X}^k \arrow[r, "\mathcal{X}^v\epsilon_u\mathcal{X}^k"] & \mathcal{X}^v\mathcal{X}^k
\end{tikzcd}$$

commutes.

Construction 8.3 $\epsilon_i : \mathcal{X}_i\mathcal{X}^i \to \mathsf{id}_{\mathsf{Lim}}$ such that $\epsilon^j_i = \mathcal{X}^j\epsilon_i$.

Proof The coherence equation H is obtained by varying the above diagram along $t : j' \to j$, which is the colimit of three commutative squares which amount to naturality of $\alpha^{t,v}$ and M for ξ. $\square$

Lemma 8.4 The diagram

$$
\begin{tikzcd}
M_i^j M_i^i \arrow[rrr, "\epsilon_i^j"] & & & M_i^j \arrow[dl, "\mu_i^v"'] \\
\mathcal{X}_u^v M_i^i \arrow[u, "\nu_u^v"] \arrow[r, "\mu_i^u"] \arrow[ur, phantom, "\mathsf{Q}"] & \mathcal{X}_u^v \mathcal{X}^u M_i^k \arrow[r, "\epsilon_u"] & \mathcal{X}^v M_i^k & \\
& \mathcal{X}_u^v \mathcal{X}_u^{u,\mathsf{id}} \arrow[u, "\nu_k^{\mathsf{id}}"] \arrow[r, "\epsilon_u"] \arrow[dl, "\alpha^{u,\mathsf{id}}"'] \arrow[ul, phantom, "\mathsf{P}"] \arrow[ur, phantom, "\mathsf{N}"] & \mathcal{X}_u^{v,\mathsf{id}} \arrow[u, "\nu_u^{\mathsf{id}}"] \arrow[dr, "\alpha^{v,\mathsf{id}}"] \arrow[ur, phantom, "\mathsf{P}"] & \\
\mathcal{X}_u^v \mathcal{X}_u^u \arrow[uu, "\nu_u^u"] \arrow[r, "\mathsf{id}"] \arrow[ur, phantom, "\mathsf{R}"] & \mathcal{X}_u^v \mathcal{X}_u^u \arrow[u, "\gamma^k"] \arrow[r, "\epsilon_u"] \arrow[ur, phantom, "\mathsf{N}"] & \mathcal{X}_u^v \arrow[u, "\gamma^k"] \arrow[r, "\mathsf{id}"] \arrow[ur, phantom, "\mathsf{R}"] & \mathcal{X}_u^v \arrow[uuu, "\nu_u^v"']
\end{tikzcd}
$$

commutes. ☐

Lemma 8.5 $\epsilon_i : \mathcal{X}_i \mathcal{X}^i \to 1_L$ is a colimiting cocone.
Proof *Cocone (over what?) Colimiting: use finality.

$$
\begin{tikzcd}
\left[\operatorname{colim}^{\uparrow}_{i \in \mathcal{I}} \mathcal{X}_i \mathcal{X}^i\right]^j \arrow[r] & \mathcal{X}_j \arrow[rr, equal] & & \mathcal{X}_j \arrow[d, "\xi^j"'] \\
\mathcal{X}^j \mathcal{X}_i \mathcal{X}^i \arrow[u] \arrow[r, equal] & \mathcal{X}^j \mathcal{X}_i \mathcal{X}^i \arrow[u, "\epsilon_i^j"'] & & \mathcal{X}^v \mathcal{X}^k \\
& \mathcal{X}^v \mathcal{X}_u \mathcal{X}^i \arrow[u, "\nu_u^v \mathcal{X}^i"'] \arrow[r, equal, "u = \mathsf{id}_k"] & \mathcal{X}^v \mathcal{X}_{\mathsf{id}} \mathcal{X}^k \arrow[r, "\mathcal{X}^v \mathcal{X}_{\mathsf{id}} \xi^{\mathsf{id}}"] & \mathcal{X}^v \mathcal{X}^{\mathsf{id}} \mathcal{X}_{\mathsf{id}} \mathcal{X}^k \arrow[u, "\mathcal{X}^v \epsilon_{\mathsf{id}} \mathcal{X}^k"]
\end{tikzcd}
$$

The details are in the diagram defining ϵ_i^j. *Cocones, *coherences. ☐

Lemma 8.6 $\eta^u \,;\, \nu_u^u : \mathsf{id}_{\mathcal{X}^i} \to M_i^i = \mathcal{X}^i \mathcal{X}_i$ is independent of the choice of $u : i \to j$.
Proof It suffices to show that $\eta^u \,;\, \nu_u^u = \eta^{id_i} \,;\, \nu_{\mathsf{id}_i}^{\mathsf{id}_i}$. This follows from definition 7.2 (of ν) with $w = \mathsf{id}$, together with the U and C equations for γ^i and δ_i. ☐

Definition 8.7 $\eta^i = \eta^u \,;\, \nu_u^u$.

Lemma 8.8 $\eta^i \mathcal{X}^i \,;\, \mathcal{X}^i \epsilon_i = \mathsf{id}_{\mathcal{X}^i}$.

Proof

$$
\begin{tikzcd}
M_i^i \mathcal{X}^i \arrow[rr, "\epsilon_i^i = \mathcal{X}^i \epsilon_i"] & & \mathcal{X}^i \arrow[dd, "\xi^{\mathsf{id}_i}"'] \\
& \mathsf{Q} & \\
\mathcal{X}^{\mathsf{id}_i} \mathcal{X}_{\mathsf{id}_i} \mathcal{X}^i \arrow[uu, "\nu_{\mathsf{id}_i}^{\mathsf{id}_i} \mathcal{X}^i"'] \arrow[r, "\mathcal{X}^{\mathsf{id}_i} \mathcal{X}_{\mathsf{id}_i} \xi^{\mathsf{id}_i}"] & \mathcal{X}^{\mathsf{id}_i} \mathcal{X}_{\mathsf{id}_i} \mathcal{X}^{\mathsf{id}_i} \mathcal{X}^i \arrow[r, "\mathcal{X}^v \epsilon_{\mathsf{id}_i} \mathcal{X}^i"] & \mathcal{X}^v \mathcal{X}^k \\
& & \\
\mathcal{X}^i \arrow[uu, "\eta^{id_i} \mathcal{X}^i"] \arrow[r, "\xi^{\mathsf{id}}"] & \mathcal{X}^u \mathcal{X}^i \arrow[uu, "\mathsf{Z}\eta^{id_i} \mathcal{X}^{\mathsf{id}_i} \mathcal{X}^i"'] \arrow[uur, "\mathsf{Y}", "\mathsf{id}"'] &
\end{tikzcd}
$$

$\square$

Lemma 8.9 $\mathcal{X}_i \eta^i = \epsilon_i \mathcal{X}_i$.

Proof By enrichment of the pseudo limit, it suffices to verify componentwise that $M_i^j \eta^i \,; \epsilon_i^j \mathcal{X}_i = \mathsf{id}_{M_i^j}$. The diagram

$$
\begin{tikzcd}
M_i^j \arrow[r, "M_i^j \eta^i"] & M_i^j M_i^i \arrow[r, "\epsilon_i^j \mathcal{X}_i"] & M_i^j \\
& \mathcal{X}_u^v M_i^i \arrow[u, "\mathsf{Z}\ \nu_u^v M_i^i"'] & \\
\mathcal{X}_u^v \arrow[uu, "\nu_u^v"] \arrow[ur, "\eta"] \arrow[r, "\eta"] & \mathcal{X}_u^v \mathcal{X}_u^u \arrow[u, "\mathcal{X}_u^v \nu_u^u"'] \arrow[r, "\mathcal{X}^v \epsilon_u \mathcal{X}_u"] & \mathcal{X}_u^v \arrow[uu, "\nu_u^v"]
\end{tikzcd}
$$

commutes for all $(u, v) \in (i, j)/\mathcal{I}$; the bottom row is the identity by X. The left-hand side is part of a colimiting cocone, and the diagram and provides another cocone for which the top row and the identity both serve as the mediator. $\square$

Lemma 8.10 $\mathsf{Lim}\, i : \mathcal{I}.\mathcal{X}^i$ is the pseudo-colimit, with cocone $\mathcal{X}_i$.

Proof Let $G_i : \mathcal{X}^i \to \mathcal{A}$ be another cocone for this diagram with coherences ... Consider the diagram of type $\mathcal{I}$ of functors $\mathsf{Lim}\, i : \mathcal{I}.\mathcal{X}^i \to \mathcal{A}$ with $i \mapsto A_i \mathcal{X}^i$ which takes $u : i \to j$ to

Functoriality... Let A be the colimit. Then

$$A\mathcal{X}_i = \operatorname*{colim}^{\uparrow}_{j} A_j M_i^j = \quad \operatorname*{colim}^{\uparrow}_{j} \operatorname*{colim}^{\uparrow}_{(u,v) \in \mathcal{I}/(i,j)} A_j \mathcal{X}^v \mathcal{X}_u \cong \operatorname*{colim}^{\uparrow}_{u : k \to i} A_k \mathcal{X}_u = \quad A_i$$

*Uniqueness. $\square$

9 Pseudolimit of adjunctions

Finally we shall show that we have the limit in the category of homomorphisms. Let $G^i : X \to \mathcal{X}^i$ be a cone of homomorphisms and $F_i \dashv G^i$ be the corresponding comparisons, with units υ^i and counits δ_i. Since L is the limit we have a unique mediating functor $G : X \to L$ with $G^i = G\mathcal{X}^i$, and similarly, since it is also the colimit, $F : L \to X$ with $F_i = F\mathcal{X}_i$.

Construction 9.1 Natural transformations $\upsilon : \mathsf{id}_L \to GF$ and $\delta : FG \to \mathsf{id}_X$, by the diagrams

$$\begin{tikzcd}
\mathsf{id}_L \arrow[r, "\upsilon"] & GF \\
\mathcal{X}_k\mathcal{X}^k \arrow[u, "\epsilon_k"] \arrow[r, "\mathcal{X}_k\upsilon^k\mathcal{X}^k"] & \mathcal{X}_kG^kF_k\mathcal{X}^k \arrow[u, "\epsilon_k GF\epsilon_k"']
\end{tikzcd}
\qquad
\begin{tikzcd}
sFG \arrow[r, "\delta"] & \mathsf{id}_X \\
F_kG^k \arrow[u, "F\epsilon_kG"] \arrow[ur, "\delta_k"'] &
\end{tikzcd}$$

Lemma 9.2 $F \dashv G$ with unit υ and counit δ.

Proof The two triangle laws follow from these diagrams:

$$\begin{tikzcd}
F \arrow[r, "F\upsilon"] & FGF \arrow[r, "\delta F"] & F \\
F_k\mathcal{X}^k \arrow[u] \arrow[r, "\mathcal{X}_k\upsilon^k\mathcal{X}^k"] & F_kG^kF_k\mathcal{X}^k \arrow[u] \arrow[r, "\delta_kF_k\mathcal{X}^k"] & F_k\mathcal{X}^k \arrow[u]
\end{tikzcd}$$

and

$$\begin{tikzcd}
G^j \arrow[r, "\mathcal{X}^j\upsilon G"] & G^jFG \arrow[r, "G^j\delta"] & G^j \\
M_i^jG^i \arrow[u] \arrow[r] & M_i^jG^iF_iG^i \arrow[u] \arrow[r] & M_i^jG^i \arrow[u] \\
\mathcal{X}^v\mathcal{X}_uG^j \arrow[u] \arrow[r, "\mathcal{X}^v\mathcal{X}_u\upsilon^iG^i"] & \mathcal{X}^v\mathcal{X}_uG^iF_iG^i \arrow[u] \arrow[r, "\mathcal{X}^v\mathcal{X}_uG^i\delta_i"] & \mathcal{X}^v\mathcal{X}_uG^i \arrow[u]
\end{tikzcd}$$

□

Theorem 9.3 For any cofiltered diagram of homomorphisms, the limit of the homomorphisms *quâ* continuous functors, the colimit of the comparisons *quâ* continuous functors and the limit of the homomorphisms *quâ* homomorphisms exist and are naturally equivalent.

Proof It only remains to formulate and prove naturality. One way of stating this is that the forgetful functors $\mathbf{FCCat}^{\mathsf{hm}} \to \mathbf{FCCat}$ and $\mathbf{FCCat}^{\mathsf{hm}} \to \mathbf{FCCat}^{\mathsf{op}}$ create limits. □

We call L the *bilimit* of the diagram.

10 Notes

Definition 8.2:

$$
\begin{tikzcd}[column sep=small]
 & \mathcal{X}^k \arrow[rr, "\xi^v"] \arrow[drr, phantom, "\mathsf{Q}"] & & \mathcal{X}^v\mathcal{X}^l \arrow[dddddd, dash, bend left=90] & \\
 & M_j^k\mathcal{X}^j \arrow[u, "\epsilon_i^k"] \arrow[d, "M_j^k\xi^u"'] \arrow[drr, phantom, "\mathsf{N}"] & & \mathcal{X}^v\mathcal{X}_{w\circ u}\mathcal{X}^j \arrow[ll, "\nu^v_{w;u}\mathcal{X}^j"'] \arrow[r, "\mathcal{X}^v\mathcal{X}_{w\circ u}\xi^u"] \arrow[d, "\mathcal{X}^v\mathcal{X}_{w\circ u}\xi^u"] \arrow[dr, phantom, "\mathsf{M}"] & \mathcal{X}^v\mathcal{X}_{w'}\mathcal{X}^{w'}\mathcal{X}^l \arrow[ul, "\mathcal{X}^v\epsilon_{w'}\mathcal{X}^l"'] \\
 & M_j^k\mathcal{X}^u\mathcal{X}^i \arrow[dr, phantom, "\mathsf{P}"] & & \mathcal{X}^v\mathcal{X}_{w\circ u}\mathcal{X}^u\mathcal{X}^i \arrow[ll, "\nu^v_{w;u}\mathcal{X}^u\mathcal{X}^i"'] \arrow[r, "\xi"] & \mathcal{X}^v\mathcal{X}_{w\circ u}\mathcal{X}^u\mathcal{X}^w\mathcal{X}^l \arrow[u, "\alpha"'] \arrow[dd, "\beta"'] \\
 \mathcal{X}^{\mathsf{id}}M_i^k\mathcal{X}_u\mathcal{X}^u\mathcal{X}_i \arrow[ur, "\mu_u^k\mathcal{X}^u\mathcal{X}^i"] \arrow[rr, "\mathcal{X}^{\mathsf{id}}\nu_u^v"] & & \mathcal{X}^{\mathsf{id}}\mathcal{X}_w^v\mathcal{X}^v\mathcal{X}_u\mathcal{X}^u\mathcal{X}_i \arrow[r, "\alpha"] \arrow[d, phantom, "\mathsf{Z}"] & \mathcal{X}^v\mathcal{X}_w\mathcal{X}_u\mathcal{X}^u\mathcal{X}^i \arrow[u, "\beta"'] \arrow[r, phantom, "\mathsf{N}"] & {} \arrow[r, phantom, "\mathsf{B}"] & {} \\
 & M_i^k\mathcal{X}_u\mathcal{X}^u\mathcal{X}^i \arrow[ul, "\gamma_k"] \arrow[uu, dotted] \arrow[rr, "\nu_u^v\mathcal{X}_u\mathcal{X}^u\mathcal{X}_i"] \arrow[drr, phantom, "\mathsf{N}"] & & \mathcal{X}^v\mathcal{X}_w\mathcal{X}_u\mathcal{X}^u\mathcal{X}^i \arrow[u, "\mathsf{id}"'] \arrow[ul, "\gamma"] \arrow[r, "\mathcal{X}^v_{wu}\mathcal{X}^{uw}\xi^w"] \arrow[dr, phantom, "\mathsf{Z}"] & \mathcal{X}^v\mathcal{X}_w\mathcal{X}_u\mathcal{X}^u\mathcal{X}^w\mathcal{X}^l \\
 & M_i^k\mathcal{X}^i \arrow[u, "M_i^k\epsilon_u\mathcal{X}^i"] \arrow[d, "\epsilon_i^k"'] \arrow[drr, phantom, "\mathsf{Q}"] & & \mathcal{X}^v\mathcal{X}_w\mathcal{X}^i \arrow[ll, "\nu_u^v\mathcal{X}^i"'] \arrow[u, "\mathcal{X}^v\epsilon_w\epsilon_u\mathcal{X}^i"] \arrow[r, "\mathcal{X}^v\mathcal{X}_w\xi^w"] & \mathcal{X}^v\mathcal{X}_w\mathcal{X}^l \arrow[u, "\mathcal{X}^v\mathcal{X}_w\epsilon_u\mathcal{X}^w\mathcal{X}^l"] \arrow[dl, "\mathcal{X}^v\epsilon_w\mathcal{X}^l"'] \\
 & \mathcal{X}^k \arrow[rr, "\xi^v"] & & \mathcal{X}^v\mathcal{X}^l &
\end{tikzcd}
$$

Construction 9.1:

$$
\begin{tikzcd}
\mathsf{id} & FG \arrow[l, "\delta"'] \\
F_kG^k = F_k\mathcal{X}^kG \arrow[u, "\delta_k"] \arrow[r] & F\mathcal{X}_kX^kG \arrow[u, "F\epsilon_kG"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{X}^i \arrow[dd, "\mathcal{X}_u"'] \arrow[dr, "A_i"] & \\
 & \mathcal{A} \arrow[l, phantom, "\sigma_u\Downarrow"] \\
\mathcal{X}^j \arrow[ur, "A_j"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=small]
\mathsf{Lim}\,\mathcal{X}^i \arrow[rrrr] \arrow[drr, "\mathcal{X}_i"] \arrow[ddrr] & & & & \mathcal{A} \\
 & \arrow[r, phantom, "\overset{\xi^u}{\Longrightarrow}"] & \mathcal{X}^i \arrow[urr, "A_i"'] \arrow[d, shift left, "\mathcal{X}_u"] & \arrow[r, phantom, "\overset{\sigma_u}{\Longrightarrow}"] & \\
 & & \mathcal{X}^j \arrow[u, shift left, "\mathcal{X}^u"] \arrow[uurr, "A_i"'] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
A_i\mathcal{X}^i \arrow[d, "\sigma_u \cdot \xi^u"'] \arrow[dr, "\tau_i"] & \\
A_j\mathcal{X}_u\mathcal{X}^u\mathcal{X}^j \arrow[d, "A_j\epsilon_u\mathcal{X}^j"'] & A \\
A_j\mathcal{X}^j \arrow[ur, "\tau_j"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & A\mathcal{X}_i \arrow[ddr, "\sigma_i"] & \\
 & A_i\mathcal{X}^i\mathcal{X}_i \arrow[u, "\tau_i\mathcal{X}_i"] \arrow[dr, "A_i\epsilon_i"] & \\
A_i \arrow[ur, "A_i\eta^i"] \arrow[r, "A_i\eta^u"] & A_i\mathcal{X}^u\mathcal{X}_u \arrow[u, "A_i\nu^u_u"] & A_i \\
A_j\mathcal{X}_u \arrow[u, "\sigma_u"] \arrow[r, "A_j\mathcal{X}_u\eta^u"] & A_j\mathcal{X}_u\mathcal{X}^u\mathcal{X}_u \arrow[u, "\sigma_u\mathcal{X}^u\mathcal{X}_u"] \arrow[r, "A_j\epsilon_u\mathcal{X}^u"] & A_j\mathcal{X}_u \arrow[u, "\sigma_u"']
\end{tikzcd}
\qquad
\begin{tikzcd}
A\mathcal{X}_i \arrow[rr, "\sigma_i", "\cong"'] & & A_i \\
 & A_i\mathcal{X}_i\mathcal{X}^i \arrow[ul, "A_i\tau_i"] \arrow[ur, "A_i\epsilon^i"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
A\mathcal{X}_j\mathcal{X}_u \arrow[r, "A\iota_u"] \arrow[d, "\sigma_j\mathcal{X}_u"'] & A\mathcal{X}_i \arrow[d, "\sigma_i"] \\
A_j\mathcal{X}_u \arrow[r, "\sigma_u"] & A_i
\end{tikzcd}
\qquad
\begin{tikzcd}
 & A & \\
A_i\mathcal{X}^i \arrow[ur, "\tau_i"] & & A\mathcal{X}_i\mathcal{X}^i \arrow[ul, "A\epsilon_i"'] \arrow[ll, "\sigma_i\mathcal{X}^i"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
A & \\
A_i\mathcal{X}^i \arrow[u, "\tau_i"] & A\mathcal{X}_i\mathcal{X}^i \arrow[ul, "A\epsilon_i"'] \arrow[l, "\sigma_i\mathcal{X}^i"'] \\
A_i\mathcal{X}_u\mathcal{X}^u\mathcal{X}^i \arrow[u, "A_i\epsilon_i\mathcal{X}^i"] \arrow[ur, phantom, "\mathsf{N}"] & A\mathcal{X}_i\mathcal{X}_u\mathcal{X}^u\mathcal{X}^i \arrow[u, "A\mathcal{X}_i\epsilon_i\mathcal{X}^i"'] \arrow[l, "\sigma_i\mathcal{X}_u\mathcal{X}^u\mathcal{X}^i"'] \\
A_j\mathcal{X}^j \arrow[u, "\sigma_u\xi^u"] \arrow[ur, phantom, "\text{pseudo} - \text{colim}"] & A\mathcal{X}_j\mathcal{X}^j \arrow[u, "A\iota_u\xi^u"'] \arrow[l, "\sigma_j\mathcal{X}^j"']
\end{tikzcd}
$$

$\upsilon^i F^i \,;\, F^i\delta_i = \mathsf{id}F^i$ $\upsilon^i : \mathsf{id}_{\mathcal{X}^i} \to F^iG_i$ $\delta_i : G_iF_i \to \mathsf{id}_A$ $\theta_i : G_i \to G\mathcal{X}_i$

$$
\begin{tikzcd}
F^i \arrow[r, "\upsilon^i F^i"] \arrow[d, "\eta^i F^i"'] \arrow[dd, bend right=60, "\mathsf{id}"'] \arrow[dr, phantom, "\mathrm{def}"] & F^i G_i F^i \arrow[r, "F^i \delta_i"] \arrow[d, "\mathcal{X}^i F \theta_i \mathcal{X}^i F"] \arrow[dr, phantom, "\mathrm{def}"] & F^i \\
\mathcal{X}^i \mathcal{X}_i F^i \arrow[r, "\mathcal{X}^i \upsilon \mathcal{X}_i \mathcal{X}^i F"] \arrow[d, "\epsilon_i F^i"'] \arrow[dr, phantom, "\mathsf{N}"] & \mathcal{X}^i F G \mathcal{X}_i \mathcal{X}^i F \arrow[r, "\mathcal{X}^i F G \epsilon_i F"] \arrow[d, "\mathcal{X}^i F G \epsilon_i F"] & \mathcal{X}^i F G F \arrow[u, "\mathcal{X}^j F \delta"'] \\
\mathcal{X}^i F \arrow[r, "\mathcal{X}^i \upsilon F"] & \mathcal{X}^i F G F \arrow[ur, "\mathsf{id}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{A} \arrow[dd, shift left, "F"] \arrow[dr, shift left, "F^i \dashv"] & \\
& \mathcal{X}^i \arrow[ul, shift left, "G_i"] \arrow[dl, shift left, "\mathcal{X}_i"] \\
\mathsf{Lim} \arrow[uu, shift left, "G"] \arrow[uu, phantom, "\dashv"] \arrow[ur, shift left, "\mathcal{X}^i \dashv"] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathsf{id}_{\mathcal{X}^i} \arrow[r, "\upsilon^i"] \arrow[d, "\eta^i"'] \arrow[dr, phantom, "\mathrm{def}"] & \mathcal{X}^i F G_i \arrow[d, "\mathcal{X}^i F \theta_i"] \\
\mathcal{X}^i \mathcal{X}_i \arrow[r, "\mathcal{X}^i \nu \mathcal{X}_i"] & \mathcal{X}^i F G \mathcal{X}_i \\
\mathcal{X}^v \mathcal{X}_u \arrow[u, "\nu_u^v"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
G_i F^i \arrow[r, "\delta_i"] \arrow[d, "\theta_i F^i = \theta_i \mathcal{X}^i F"] \arrow[dr, phantom, "\mathrm{def}"] & \mathsf{id}_{\mathcal{A}} \\
G \mathcal{X}_i F^i = G \mathcal{X}_i \mathcal{X}^i F \arrow[r, "G \epsilon_i F", "\text{colimiting}"'] & GF \arrow[u, dotted, "\delta"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
G_i \arrow[r, "G_i \upsilon^i"] \arrow[d, "\theta_i"'] \arrow[dr, phantom, "\mathsf{Z}"] & G_i F^i G_i \arrow[r, "\delta_i G_i"] \arrow[d, "\theta_i F^i G_i"] \arrow[dr, phantom, "\mathrm{def}"] & G_i \arrow[ddd, bend left=60, "\theta_i"] \\
G \mathcal{X}_i \arrow[r, "G \mathcal{X}_i \upsilon^i"] \arrow[d, "G \mathcal{X}_i \eta^i"'] \arrow[dd, bend right=60] \arrow[dr, phantom, "\mathrm{def}"] & G \mathcal{X}_i \mathcal{X}^i F G_i \arrow[r, "G \epsilon_i F G_i"] \arrow[d, "G \mathcal{X}_i \mathcal{X}^i F_i \theta_i"] \arrow[dr, phantom, "\mathsf{N}"] & G F G_i \arrow[u, "\delta G_i"'] \arrow[d, "G F \theta_i"] \arrow[d, phantom, shift left=8, "\mathsf{N}"] \\
G \mathcal{X}_i \mathcal{X}^i \mathcal{X}_i \arrow[r, "G \mathcal{X}_i \mathcal{X}^i \upsilon \mathcal{X}_i"] \arrow[d, "G \epsilon_i \mathcal{X}_i"'] \arrow[dr, phantom, "\mathsf{Z}"] & G \mathcal{X}_i \mathcal{X}^i F G \mathcal{X}_i \arrow[r, "G \epsilon_i F G \mathcal{X}_i"] \arrow[d, "G \epsilon_i F G \mathcal{X}_i"] & G F G \mathcal{X}_i \arrow[d, "\delta G \mathcal{X}_i"] \\
G \mathcal{X}_i \arrow[r, "G \upsilon \mathcal{X}_i"] & G F G \mathcal{X}_i \arrow[r, "\delta G \mathcal{X}_i"] & G \mathcal{X}_i
\end{tikzcd}
$$